Name SOLUTIONS

March 24, 2009

Math 133 — Hourly Exam 3B

Do all problems (Total = 100 points). Show all work. Calculators are not allowed.

(1) (20 points) Compute the following integrals. *Make your method and substitutions clear.*

(a) $\int \tan x \sec^4 x \, dx = \int u^3 \, du = \frac{u^4}{4} + C$

$u = \sec x$
$du = \sec x \tan x \, dx$

$= \frac{1}{4} \sec^4 x + C$

(b) $\int \sin^2 x \cos^2 x \, dx = \int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} \, dx.$

$= \frac{1}{4} \int 1 - \cos^2(2x) \, dx.$

$= \frac{x}{4} - \frac{1}{4} \int \frac{1 - \cos(4x)}{2} \, dx.$

$= \frac{x}{4} - \frac{x}{8} + \frac{1}{8} \frac{\sin(4x)}{4} + C$

$= \frac{3}{8} x + \frac{1}{32} \sin(4x) + C$

(2) (10 points) Evaluate the improper integral $I = \int_2^3 \frac{x \, dx}{\sqrt{9 - x^2}}$. *Show all steps.*

With $\left[\begin{array}{l} u = 9 - x^2 \\ du = -2x \, dx. \\ -\frac{1}{2} du = x \, dx \end{array}\right]$. $\int \frac{x \, dx}{\sqrt{9-x^2}} = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\frac{1}{2} \int u^{-1/2} = -\frac{1}{2} 2 u^{1/2} + C$

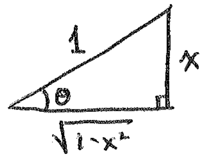
$= -\sqrt{9-x^2} + C$

Since the integrand has an asymptote at $x = 3$, the integral I is:

$I = \lim_{a \to 3^-} \int_2^a \frac{x \, dx}{\sqrt{9-x^2}} = \lim_{a \to 3^-} -\sqrt{9-x^2} \Big|_2^a$

$= \lim_{a \to 3^-} -\left(\sqrt{9-x^2} - \sqrt{5}\right) = \sqrt{5}$

(3) (10 points) Find $I = \displaystyle\int \frac{8x^2\, dx}{\sqrt{1-x^2}}$.



$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\sqrt{1-x^2} = \cos\theta$

$$I = 8\int \frac{\sin^2\theta \cdot \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= 8\int \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= 4\theta - 4\,\frac{\sin(2\theta)}{2} + C$$

$$= 4\theta - 2\cdot 2\sin\theta\cos\theta + C$$

$$= \underline{4\arcsin x - 4x\sqrt{1-x^2} + C}$$

(3) Determine whether the following integrals converge or diverge. DO NOT EVALUATE. Justify your answers by the Direct Comparison Test (DCT) or Limit Comparision Test (LCT).

(a)(8 pts) $I = \displaystyle\int_1^\infty \frac{dx}{x\sqrt{x+3}}$

<u>Method 1</u> Since $x+3 \geq x$ we have $\frac{1}{x\sqrt{x+3}} \leq \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$. But $\int_1^\infty \frac{1}{x^{3/2}}\, dx$ converges by the p-test, so <u>I converges by the DCT.</u>

<u>Method 2</u> With $f(x) = \frac{1}{x\sqrt{x+3}}$ and $g(x) = \frac{1}{x\sqrt{x}}$ we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{x}}{\cancel{x}\sqrt{x+3}} = \lim_{x\to\infty}\sqrt{\frac{x}{x+3}} = \sqrt{\lim_{x\to\infty} \frac{1}{1+3/x}} = 1.$$

Since $\int_1^\infty g(x) = \int_1^\infty \frac{dx}{x^{3/2}}$ converges by the p-test, <u>I converges by the LCT.</u>

(b)(6 pts) $I = \int_{10}^{\infty} \frac{\ln x}{x}\, dx.$ Because $\ln x \geq 1$ for all $x \geq 10$ we know that

$\frac{\ln x}{x} \geq \frac{1}{x}$ and $\int_{10}^{\infty} \frac{1}{x}\, dx$ diverges (by the p-test).

$\Rightarrow$ $\underline{I \text{ diverges by the DCT}}$

(4) (14 points) Solve the differential equation $\frac{\cos y}{x} - \sin y \cdot \frac{dy}{dx} = 0$ for $0 \leq x \leq \frac{\pi}{2}$ with the initial condition $y(2) = 0$.

Separate variables: $\frac{dx}{x} = \frac{\sin y}{\cos y}\, dy.$

Integrate: $\int \frac{dx}{x} = \int \tan y\, dy$

$$\ln |x| = -\ln|\cos y| + c$$

$$x = e^{-\ln|\cos y| + c} = \frac{1}{|\cos y|} e^{c} = \frac{C}{|\cos y|}$$

new constant

$$\Rightarrow \cos y = \pm \frac{C}{x}$$

Plug in $x=2, y=0$ $\quad 1 = \pm\frac{C}{2} \Rightarrow C = \pm 2$

Solve for y: $\boxed{y = \arccos(\pm 2/x)}$ (two solutions)

(5) (9+9+6 points) Calculate the limit of the following sequences. *Show all steps clearly.*

(a) $\lim_{n\to\infty} \frac{\ln\left(\frac{1}{n}\right)}{\sqrt{2n}} = \lim_{n\to\infty} \frac{-\ln n}{\sqrt{2}\sqrt{n}} = -\frac{1}{\sqrt{2}} \lim_{x\to\infty} \frac{\ln x}{x^{1/2}}$ type $\frac{\infty}{\infty}$

$$\overset{\text{L'Hop}}{=} -\frac{1}{\sqrt{2}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}}$$

$$= -\frac{2}{\sqrt{2}} \lim_{x\to\infty} \frac{\sqrt{x}}{x}$$

$$= -\sqrt{2} \lim_{x\to\infty} \frac{1}{\sqrt{x}} = \boxed{0}$$

(b) $\lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^n = \lim_{n\to\infty} e^{\ln\left(1-\frac{2}{n+1}\right)\cdot n}$

$$= e^{\lim_{n\to\infty} \ln\left(1-\frac{2}{n+1}\right)\cdot n}$$

switch letters and write n as $\frac{1}{\frac{1}{n}}$.

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x+1}\right)}{1/x}} \quad \text{type } \frac{\infty}{\infty}$$

$$\underline{\underline{\text{L'Hop}}}\ e^{\lim_{x\to\infty} \frac{\frac{1}{1-2/x+1}\cdot\frac{+2}{(x+1)^2}}{-1/x^2}} = e^{\left(\lim_{x\to\infty}\frac{1}{1-2/x+1}\right)\left(\lim_{x\to\infty}\frac{-2x^2}{(x+1)^2}\right)}$$

$$= e^{-2\cdot\left(\lim_{x\to\infty}\frac{x}{x+1}\right)^2}$$

$$= e^{-2} = \boxed{\frac{1}{e^2}}$$

(c) If $a_n = \dfrac{1+\sin n}{n}$ what is $\lim_{n\to\infty} a_n$? Justify your answer.

Since $-1 \le \sin n \le 1$ for any number n, we have $0 \le 1+\sin n \le 2$ and

$$\frac{-1}{n} \le \frac{1+\sin n}{n} \le \frac{1}{n}$$

But $\lim_{n\to\infty}\frac{1}{n}=0$ and $\lim_{n\to\infty}\frac{-1}{n}=0$, so $\lim_{n\to\infty} a_n = 0$ by the sandwich theorem.

(6) (8 pts) For the series below, write down the first 4 terms and calculate the sum:

$$\sum_{n=1}^{\infty}\left(\frac{-1}{5}\right)^{n-2} = \left(\frac{-1}{5}\right)^{-1} + \left(\frac{-1}{5}\right)^{0} + \left(\frac{-1}{5}\right)^{1} + \left(\frac{-1}{5}\right)^{2} + \cdots$$

$$= -5 + 1 - \frac{1}{5} + \frac{1}{25} - \cdots$$

This is a geometric series with $A=-5$ and $r=-1/5$. Because $|r| = |1/5| < 1$ it converges to

$$\frac{A}{1-r} = \frac{-5}{1-(-1/5)} = \frac{-5}{6/5} = \boxed{-\frac{25}{6}}$$