

## Math 133 — Hourly Exam 4A

Do all problems (Total = 100 points). Show all work.

(1) (30 points) Determine if the following positive series converge and state which test confirms your answer ( $n^{\text{th}}$  term, Integral and Ratio tests, DCT, LCT, AST) *Justify your answers* in the space provided.

$$(a) S = \sum_{n=1}^{\infty} \frac{3n}{\sqrt{2n^5 + n^2 + 1}} = \sum_{n=1}^{\infty} a_n \quad \text{Compare with } b_n = \frac{3n}{\sqrt{2n^5}} = \frac{3}{\sqrt{2}} \cdot \frac{1}{n^{3/2}}$$

Note that  $\sum b_n = \frac{3}{\sqrt{2}} \sum \frac{1}{n^{3/2}}$  converges (p-series).

Method 1 For large  $n$   $\sqrt{2n^5 + n^2 + 1} > \sqrt{2n^5}$ , so  $a_n < b_n$ . Since  $\sum b_n$  converges,  $S$  converges by the DCT.

Method 2.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{2n^5 + n^2 + 1}} \cdot \frac{\sqrt{2} n^{3/2}}{3} = \lim_{n \rightarrow \infty} \sqrt{\frac{2n^5}{2n^5 + n^2 + 1}} = \sqrt{\lim_{n \rightarrow \infty} \frac{2}{2 + \frac{1}{n^3} + \frac{1}{n^5}}} = 1$

Since  $\sum b_n$  converges,  $S$  converges by the LCT.

$$(b) S = \sum_{n=1}^{\infty} \frac{n}{2n-5}$$

$\lim_{n \rightarrow \infty} \frac{n}{2n-5} = \lim_{n \rightarrow \infty} \frac{1}{2 - 5/n} = \frac{1}{2}$ . This limit is not 0, so  $S$  diverges by the  $n^{\text{th}}$  term test.

(c)  $S = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$  By the Integral TEST, this converges or diverges according to whether  $\int_3^{\infty} \frac{dx}{x(\ln x)^2}$  is finite. Set  $\begin{cases} u = \ln x \\ du = \frac{dx}{x} \end{cases}$

$$= \int_{\ln 3}^{\infty} \frac{du}{u^2} = \lim_{L \rightarrow \infty} \int_{\ln 3}^L u^{-2} du = \lim_{L \rightarrow \infty} \left. -\frac{1}{u} \right|_{\ln 3}^L = 0 + \ln 3 \quad \text{finite.}$$

$\Rightarrow S$  converges by I Test.

(2) (10 points) Determine if the series below is absolutely convergent (AC), conditionally convergent (CC), or divergent. Justify your answer and make clear which test(s) you use.

$$S = \sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!} \quad \text{Ratio Test with } a_n = \frac{n!}{(2n)!} :$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{[2(n+1)]!} \cdot \frac{2n!}{n!} = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot \cancel{n!} \cdot 2n!}{(2n+2)(2n+1)\cancel{2n!} \cdot n!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)}{2(n+1) \cdot 2n+1} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0. \end{aligned}$$

$\Rightarrow$  this series converges absolutely by the Ratio Test.

(3) (10 points) What are all values of  $x$  for which the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2 5^n}$$

converges? Be sure to check for convergence at the endpoints. Write the letter of your answer: A

(A)  $-2 \leq x \leq 8$  (B)  $-2 < x \leq 8$  (C)  $-2 \leq x < 8$  (D)  $-5 \leq x \leq 5$  (E)  $-5 \leq x < 5$

(F)  $-\frac{3}{5} \leq x \leq \frac{3}{5}$  (G)  $-\frac{3}{5} < x \leq \frac{3}{5}$  (H)  $-\frac{3}{5} \leq x < \frac{3}{5}$  (I)  $-\frac{5}{3} \leq x \leq \frac{5}{3}$  (J)  $-\frac{5}{3} \leq x < \frac{5}{3}$

$$\text{Ratio Test with } a_n = \frac{|x-3|^n}{n^2 5^n} : \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{|x-3|^{n+1}}{(n+1)^2 \cdot 5^{n+1}} \cdot \frac{n^2 \cdot 5^n}{|x-3|^n}$$

$$= \lim_{n \rightarrow \infty} \frac{\cancel{|x-3|^n} \cdot |x-3|}{|x-3|^n} \cdot \lim_{n \rightarrow \infty} \frac{\cancel{5^n}}{\cancel{5^n} \cdot 5} \cdot \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2}$$

$$= \frac{|x-3|}{5} \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^2 = \frac{|x-3|}{5} \left[ \lim_{n \rightarrow \infty} \frac{1}{1+1/n} \right]^2 = \frac{|x-3|}{5}. \quad \text{So the series}$$

$$\text{Converges absolutely if } \frac{|x-3|}{5} < 1 \Rightarrow |x-3| < 5 \Rightarrow -5 < x-3 < 5 \\ -2 < x < 8$$

ENDPOINTS : When  $x=8$ , series is  $\sum \frac{5^n}{n^2 5^n} = \sum \frac{1}{n^2}$  Converges  
 When  $x=-2$  " "  $\sum \frac{(-5)^n}{n^2 5^n} = \sum \frac{(-1)^n}{n^2}$  Converges

$\Rightarrow$  Answer is (A)

(4) (a) (6 points) Rewrite the polar equation  $r = 2 \cos \theta$  in Cartesian (i.e.  $xy$ -) coordinates.

Multiply both sides by  $r$ :

$$r^2 = 2(r \cos \theta)$$

$$\boxed{x^2 + y^2 = 2x}$$

or  $y^2 = 2x - x^2$  or  $y = \pm \sqrt{2x - x^2}$ .

(5) (12 points) Find the degree 2 Taylor Polynomial  $P_2(x)$  of  $f(x) = \frac{1}{\sqrt{x}}$  about  $a = 4$ . Write the coefficients as fractions (not decimals).

$n$	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\frac{1}{\sqrt{x}} = x^{-1/2}$	$\frac{1}{\sqrt{4}} = \frac{1}{2}$
1	$-\frac{1}{2} x^{-3/2}$	$-\frac{1}{2} \cdot \frac{1}{4\sqrt{4}} = -\frac{1}{16}$
2	$-\frac{1}{2} \cdot \frac{3}{2} x^{-5/2}$	$-\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{4 \cdot 4 \cdot \sqrt{4}} = \frac{3}{128}$

$$\boxed{P_2(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2}$$

$$\frac{3}{128} = \frac{1}{2!} = \frac{3}{256}$$

(6) (10 points) Find the Maclaurin series for the function below. You are free to use the well-known series for sine. Express your answer in summation notation, that is, in the form  $\sum a_n x^n$ .

$$\boxed{x \cos \sqrt{x} =}$$

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$$

Take  $u = \sqrt{x}$

$$\cos(\sqrt{x}) = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \dots$$

$$= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots$$

Multiply by  $x$ :

$$x \cos \sqrt{x} = x - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^4}{6!} + \dots$$

Write in summation notation:

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(2n)!}$$

$$\left[ \text{or } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{(2n-2)!} \right]$$

(7) (10 points) A particle moves in the plane along the path given parametrically by

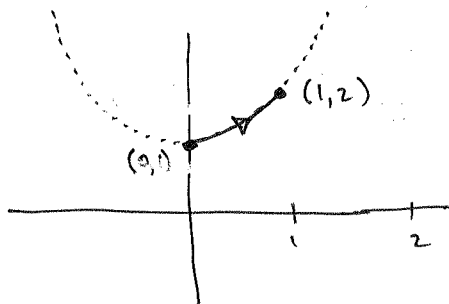
$$x(t) = \tan t, \quad y(t) = \sec^2 t.$$

(a) Find the equation of the particle's path in  $xy$  coordinates.

$$y = \sec^2 t = 1 + \tan^2 t \\ = 1 + x^2$$

$$\boxed{y = 1 + x^2}$$

(b) Graph the path of the particle for  $0 \leq t \leq \frac{\pi}{4}$ . Label the beginning and end points and include an arrow showing the direction of motion.



$$\text{When } t = 0, \quad (x, y) = (\tan 0, \sec^2 0) = (0, 1)$$

$$\text{When } t = \pi/4, \quad (x, y) = (\tan \pi/4, \sec^2 \pi/4) = (1, 2)$$

(8) (6+4+2 points) (a) Find the Maclaurin series for the function  $f(t) = \frac{1}{1+t^2}$ . You are free to use any of the well-known series that you learned.

$$\text{In geometric series } \frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

$$\text{Take } u = -t^2$$

$$\frac{1}{1+t^2} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \dots$$

$$\boxed{\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \dots}$$

(b) Use your answer to part (a) to find the Maclaurin series for  $\arctan x$  (Hint:  $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ )

$$\arctan x = \int_0^x \frac{dt}{1+t^2}$$

Fund. thm of Calc.

$$= \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \dots)$$

part a)

$$= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots \Big|_0^x$$

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots}$$

(c) Use the fact that  $\arctan 1 = \frac{\pi}{4}$  to obtain write  $\pi$  as the sum of an infinite series.

Take  $x=1$  in above to get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Multiply by 4

$$\boxed{\pi = 4 \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right]}$$