Name Solutions April 21, 2009

Math 133 — Hourly Exam 4B

Do all problems (Total = 100 points). Show all work.

(1) (30 points) Determine if the following positive series converge and state which test confirms your answer (n^{th} term, Integral and Ratio tests, DCT, LCT, AST) *Justify your answers* in the space provided.

(a) $S = \sum_{n=1}^{\infty} \frac{3n}{\sqrt{2n^5+n^2+1}}$ $= \sum_{n=1}^{\infty} a_n$ Compare with $b_n = \frac{3n}{\sqrt{2n^5}} = \frac{3}{\sqrt{2}} \cdot \frac{1}{n^{3/2}}$

Note that $\sum b_n = \frac{3}{\sqrt{2}} \sum \frac{1}{n^{3/2}}$ converges (p-series).

Method 1 For large n $\sqrt{2n^5+n^2+1} > \sqrt{2n^5}$, so $a_n < b_n$. Since $\sum b_n$ converges, S converges by the DCT.

Method 2. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{3}n}{\sqrt{2n^5+n^2+1}} \cdot \frac{\sqrt{2}\, n^{3/2}}{\cancel{3}} = \lim_{n\to\infty} \sqrt{\frac{2n^5}{2n^5+n^2+1}} = \sqrt{\lim_{n\to\infty} \frac{2}{2+\frac{1}{n^3}+\frac{1}{n^5}}} = 1$

Since $\sum b_n$ converges, S converges by the LCT

(b) $S = \sum_{n=1}^{\infty} \frac{n}{2n-5}$

$\lim_{n\to\infty} \frac{n}{2n-5} = \lim_{n\to\infty} \frac{1}{2-5/n} = \frac{1}{2}$. This limit is not 0, so S diverges by the n^{th} term test.

(c) $S = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ By the Integral Test, this converges or diverges according to whether $\int_3^{\infty} \frac{dx}{x(\ln x)^2}$ is finite. Set $\left[\begin{array}{l} u = \ln x \\ du = \frac{dx}{x}. \end{array} \right.$

$$= \int_{\ln 3}^{\infty} \frac{du}{u^2} = \lim_{L\to\infty} \int_{\ln 3}^{L} u^{-2}\, du$$

$$= \lim_{L\to\infty} -\frac{1}{u}\Big|_{\ln 3}^{L} = 0 + \ln 3 \quad \text{finite.}$$

$\Rightarrow$ S converges by I Test.

(2) (10 points) Determine if the series below is absolutely convergent (AC), conditionally convergent (CC), or divergent. *Justify* your answer and make clear which test(s) you use.

$$S = \sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$$

Ratio Test with $a_n = \frac{n!}{(2n)!}$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{[2(n+1)]!} \cdot \frac{2n!}{n!} = \lim_{n\to\infty} \frac{(n+1)\cdot \cancel{n!} \cdot \cancel{2n!}}{(2n+2)(2n+1)\cancel{(2n)!} \cdot \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}}{2\cancel{(n+1)} \cdot 2n+1}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

$\Rightarrow$ this series converges absolutely by the Ratio Test.

(3) (10 points) What are all values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2 5^n}$$

converges? *Be sure to check for convergence at the endpoints.* Write the letter of your answer: F

(A) $-\frac{3}{5} \le x \le \frac{3}{5}$ (B) $-\frac{3}{5} < x \le \frac{3}{5}$ (C) $-\frac{3}{5} \le x < \frac{3}{5}$ (D) $-\frac{5}{3} \le x \le \frac{5}{3}$ (E) $-\frac{5}{3} \le x < \frac{5}{3}$

(F) $-2 \le x \le 8$ (G) $-2 < x \le 8$ (H) $-2 \le x < 8$ (I) $-5 \le x \le 5$ (J) $-5 \le x < 5$

Ratio Test with $a_n = \frac{\quad}{n^2 5^n}$: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-3|}{(n+1)^2 \cdot 5^{n+1}} \cdot \frac{n^2 5}{|x-3|^n}$

$$= \lim_{n\to\infty} \frac{\cancel{|x-3|^n} \cdot |x-3|}{\cancel{|x-3|^n}} \cdot \lim_{n\to\infty} \frac{\cancel{5^n}}{\cancel{5^n}\cdot 5} \cdot \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \frac{|x-3|}{5} \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \frac{|x-3|}{5} \left[\lim_{n\to\infty} \frac{1}{1+1/n}\right]^2 = \frac{|x-3|}{5}.$$ So the series

converges absolutely if $\frac{|x-3|}{5} < 1 \Rightarrow |x-3| < 5 \Rightarrow -5 < x-3 < 5$

$-2 < x < 8$

<u>ENDPOINTS</u> : When $x = 8$, series is $\sum \frac{5^n}{n^2 5^n} = \sum \frac{1}{n^2}$ converges

When $x = -2$ " $\sum \frac{(-5)^n}{n^2 5^n} = \sum \frac{(-1)^n}{n^2}$ converges

$\Rightarrow$ Answer is (A)

(4) (a) (6 points) Rewrite the polar equation $r = 2\cos\theta$ in Cartesian (i.e. xy-)coordinates.

Multiply both sides by r: $r^2 = 2(r\cos\theta)$

$$\boxed{x^2 + y^2 = 2x.}$$

or $y^2 = 2x - x^2$ or $y = \pm\sqrt{2x - x^2}$.

(5) (12 points) Find the degree 2 Taylor Polynomial $P_2(x)$ of $f(x) = \dfrac{1}{\sqrt{x}}$ about $a = 4$. Write the coefficients as *fractions* (not decimals).

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\frac{1}{\sqrt{x}} = x^{-1/2}$	$\frac{1}{\sqrt{4}} = \frac{1}{2}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{4\sqrt{4}} = -\frac{1}{16}$
2	$-\frac{1}{2}\cdot\frac{-3}{2}x^{-5/2}$	$-\frac{1}{2}\cdot\frac{-3}{2}\cdot\frac{1}{4\cdot4\cdot\sqrt{4}} = \frac{3}{128}$

$$\boxed{P_2(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2.}$$

$$\frac{3}{128}\cdot\frac{1}{2!} = \frac{3}{256}$$

(6) (10 points) Find the Maclaurin series for the function below. You are free to use the well-known series for sine. *Express your answer in summation notation, that is, in the form* $\sum a_n x^n$.

$$\boxed{x\cos\sqrt{x} =}$$

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$$

Take $u = \sqrt{x}$

$$\cos(\sqrt{x}) = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots$$

$$= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots$$

Multiply by x:

$$\underline{x\cos\sqrt{x}} = x - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^4}{6!} + \cdots$$

Write in summation notation:

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(2n)!}$$

$$\left[\text{or } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{(2n-2)!}\right]$$

(7) (10 points) A particle moves in the plane along the path given parametrically by

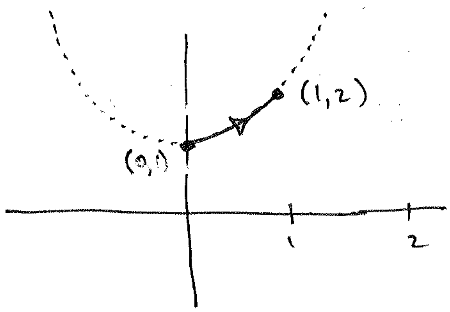
$$x(t) = \tan t, \quad y(t) = \sec^2 t.$$

(a) Find the equation of the particle's path in xy coordinates.

$y = \sec^2 t = 1 + \tan^2 t$
$= 1 + x^2$

$\boxed{y = 1 + x^2}$

(b) Graph the path of the particle for $0 \le t \le \frac{\pi}{4}$. Label the beginning and end points and include an arrow showing the direction of motion.



When $t = 0$, $(x,y) = (\tan 0, \sec^2 0) = (0,1)$

When $t = \pi/4$, $(x,y) = (\tan \frac{\pi}{4}, \sec^2 \frac{\pi}{4}) = (1,2)$

(8) (6+4+2 points) (a) Find the Maclaurin series for the function $f(t) = \frac{1}{1+t^2}$. *You are free to use any of the well-known series that you learned.*

In geometric series $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$

Take $u = -t^2$

$\frac{1}{1+t^2} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \cdots$

$\boxed{\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots}$

(b) Use your answer to part (a) to find the Maclaurin series for arctan x (Hint: $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$)

$\arctan x = \underbrace{\int_0^x \frac{dt}{1+t^2}}_{\text{Fund. thm of Calc.}} \underbrace{=}_{\text{part a)}} \int_0^x 1 - t^2 + t^4 - t^6 + t^8 + \cdots$

$= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots \Big|_0^x$

$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots}$

(c) Use the fact that $\arctan 1 = \frac{\pi}{4}$ to obtain write π as the sum of an infinite series.

Take $x = 1$ in above to get

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Multiply by 4

$\boxed{\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]}$