Partial Fractions

Method for finding integrals $\int \frac{p(x)}{q(x)}\,dx$ where $p(x)$ and $q(x)$ are polynomials:

(1) **If deg** $p \geq$ **deg** q**, DIVIDE** to get

$$\int \frac{p(x)}{q(x)} = \int \textit{polynomial} + \int \frac{r(x)}{q(x)}.$$

(2) **Factor the denominator** q into linear and irreductible quadratic factors. (Recall that a quadratic $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$).

(3) **Write** $\frac{p(x)}{q(x)}$ **as a sum** using the following rules:

- distinct linear factors : $$\frac{p(x)}{(x-a)(x-b)\ldots} = \frac{A}{x-a} + \frac{B}{x-b} + \ldots$$
- powers of linear factors: $$\frac{p(x)}{(x-a)^3} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$
- quadratic factors: $$\frac{p(x)}{(ax^2+bx+c)} = \frac{A(2ax+b)+B}{(ax^2+bx+c)}$$
- powers of quadratic factors: $$\frac{p(x)}{(ax^2+bx+c)^2} = \frac{A(2ax+b)+B}{(ax^2+bx+c)} + \frac{C(2ax+b)+D}{(ax^2+bx+c)^2}.$$

The total number of unknowns $A, B, \ldots$ should equal the degree of the denominator.

(4) **Solve for** $A, B, \ldots$ by finding a common denominator, matching numerators, and then solving by some combination of:

- plug in $x =$ some root
- plug in $x =$ some other convenient number
- matching coefficients of equal powers of x.

(5) **Integrate** using

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|, \qquad \int \frac{2ax+b}{ax^2+bx+c}\,dx = \frac{1}{a}\ln|ax^2+bx+c|, \qquad \int \frac{dx}{x^2+c^2} = \frac{1}{c}\arctan\left(\frac{x}{c}\right)$$