Solutions

Math 133 Final Exam, Fall, 2001

Name	
I.D. No.	
Section Number	

YOU MUST SHOW ALL YOUR WORK. ANSWERS WITHOUT SUPPORTING WORK WILL NOT BE ACCEPTED. CALCULATORS ARE NOT ALLOWED. THERE ARE 14 PROBLEMS. YOU SHOULD CHECK THAT YOUR EXAM HAS ALL 14 PROBLEMS.

1. (18 pts) Compute $\frac{dy}{dx}$. You do not need to simplify your answer.

(a) $y = \tan^{-1}(3x)$

$$\frac{dy}{dx} = \frac{3}{1+9x^2}$$

answer= ______________

(b) $y = \left(\ln(x^2+1)\right)^3$

$$\frac{dy}{dx} = 3\left(\ln(x^2+1)\right)^2 \cdot \frac{1}{x^2+1} \cdot 2x$$

answer= ______________

(c) $y = x^{\sin^{-1} x}$

$$\ln(y) = \sin^{-1}x \ln x$$

$$\frac{1}{y}\,dy = \left(\sin^{-1}x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{\sqrt{1-x^2}}\right)dx$$

answer= ______________

$$\frac{dy}{dx} = y\left(\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = x^{\sin^{-1}x}\left(\frac{\sin^{-1}(x)}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)$$

2. (10 pts) Solve the initial value problem $\frac{dy}{dx} = xy - x, \quad y(2) = 0.$

$= x(y-1)$

$\int \frac{1}{y-1}\, dy = \int x\, dx$

$\ln|y-1| = \frac{x^2}{2} + c \Rightarrow 0 = 2 + c \qquad c = -2$

$|y-1| = \cancel{k}e^{\frac{x^2}{2}-2}$

$y - 1 = \pm e^{x^2/2 - 2}$

$y = 1 - e^{x^2/2 - 2}$

answer=__________________________

3. (32 pts) Evaluate the integrals.

(a) $\int \underbrace{5^{\sqrt{x}}}\,\underbrace{\sqrt{x}}\,\underbrace{dx} \overset{①}{=} \int \underbrace{5^u} \cdot \underbrace{u} \cdot \underbrace{2u\, du} = 2\int u^2 5^u\, du$

$\overset{②}{=} 2\left[u^2\left(\frac{1}{\ln 5}\right)^2 5^u - 2u\left(\frac{1}{\ln 5}\right)^3 5^u + 2\left(\frac{1}{\ln 5}\right)^4 5^u\right] + C$

answer=__________________________

(b) $\int \frac{dx}{\sqrt{9 - x^2}}\, dx$

$= \sin^{-1}\left(\frac{x}{3}\right) + C$

answer=__________________________

① $u = \sqrt{x}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\, dx$

$2\sqrt{x}\, du = dx$

$2u\, du = dx$

②

$\frac{d}{du}\, u^2$		$\int 5^u$
u^2	$+$	$\left(\frac{1}{\ln 5}\right) 5^u$
$2u$	$-$	$\left(\frac{1}{\ln 5}\right)^2 5^u$
2	$+$	$\left(\frac{1}{\ln 5}\right)^3 5^u$
0		$\left(\frac{1}{\ln 5}\right)^4 5^u$

(c) $\int x \ln x \, dx$

$u = \ln x \qquad v = \frac{x^2}{2}$

$du = \frac{1}{x} dx \qquad dv = x\, dx$

$= uv - \int v \, du$

$= \left(\frac{x^2}{2}\right) \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \left(\frac{x^2}{2}\right) \ln x - \frac{1}{2} \int x \, dx = \left(\frac{x^2}{2}\right) \ln x - \left(\frac{1}{4}\right) x^2 + C$

answer=____________________________

(d) $\int \frac{18}{x^3 - 9x} dx.$ $\overset{①}{=} \int \frac{18}{x(x^2-9)} dx$

$= \int \frac{18}{3\sec\theta (9\tan^2\theta)} \cdot 3\sec\theta \tan\theta \, d\theta = 2\int \frac{\cos\theta}{\sin\theta} d\theta = 2\ln|\cos\theta| + \ldots$

$\overset{②}{=} 2\ln\left|\frac{3}{x}\right| + C$

answer=____________________________

~~4. (10 pts) A particle moves in a straight line with velocity $v(t) = t^2 - 3t + 2$. Find the total distance traveled from time $t = 0$ to time $t = 2$.~~

① $x = 3\sec\theta$

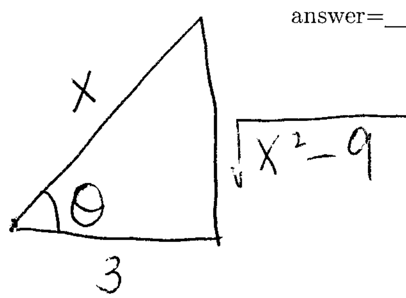
$dx = 3\sec\theta \tan\theta \, d\theta$

$x^2 - 9 = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta.$

answer=____________________________

② $\frac{x}{3} = \sec\theta$

$\cos\theta = \frac{3}{x}$



5. (7 pts) Determine whether the following improper integral converges or diverges: $\int_1^3 \frac{dx}{(3-x)^{1/2}}$.

$$\lim_{a\to 3}\int_1^a (3-x)^{-1/2}\,dx = \lim_{a\to 3}\left[-2(3-x)^{1/2}\Big|_1^a\right]$$

$$= \lim_{a\to 3} -2\left(\sqrt{3-a}\right) + 2\sqrt{3-1} = 2\sqrt{2}$$

Write converges or write diverges: converge.

6. (10 pts) Find the length of the curve $y = 2x^{3/2}$ from $x = 0$ to $x = 1$.

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+9x}\,dx = \int_0^1 (1+9x)^{1/2}\,dx$$

$$= \frac{1}{9}\cdot\frac{2}{3}(1+9x)^{3/2}\Big|_0^1 = \left(\frac{2}{27}(10)^{3/2}\right) - \left(\frac{2}{27}\right)$$

length=

7. (14 pts) Evaluate the following limits:

(a) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\quad \frac{\infty}{0} \overset{H}{=} \lim_{x\to\infty} \frac{\sin(x)}{2x} \overset{H}{=} \lim_{x\to\infty} \frac{\cos(x)}{2} = \frac{1}{2}.$

~~because~~

lim=

(b) $\lim_{x\to\infty} (\ln x)^{1/x}$.

$$= \lim_{x\to\infty} e^{\ln\left((\ln x)^{1/x}\right)} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(\ln(x))} = \text{type } \frac{\infty}{\infty} = \lim_{x\to\infty} e^{\frac{\ln(\ln(x))}{x}}$$

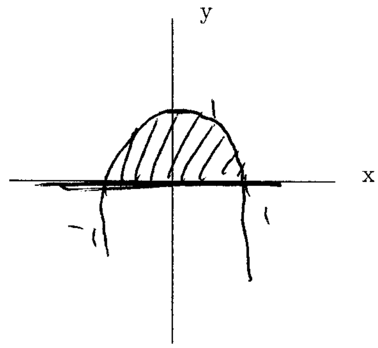
lim=

$$\overset{H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}} = e^0 = 1$$

8. (18 pts) Consider the region R enclosed by $y = 1 - x^2$ and $y = 0$.

$y=0$ is the x-axis.

(a) Sketch the region.



Set up completely, but DO NOT COMPUTE, an integral (or integrals) for the volume of the solid produced when

(b) The region R is rotated about the x-axis.

$$\cancel{V} = \int_{-1}^{1} \pi (1-x^2)^2 \, dx.$$

answer=

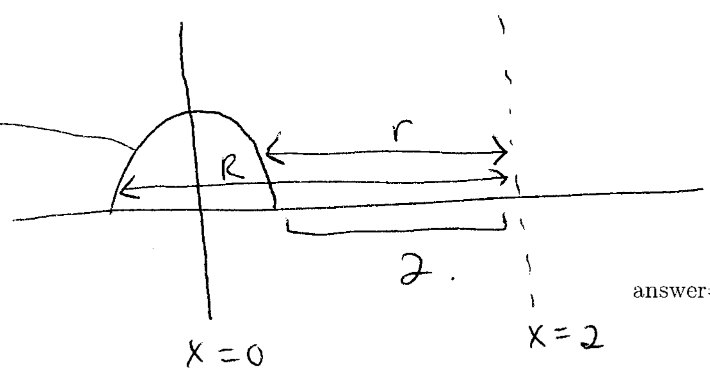
(c) R is rotated about the line $x = 2$.

Label by y, $0 \le y \le 1$

$y = 1-x^2$

$x^2 = 1-y$

$x = \sqrt{1-y}$.



$r = 2 - \sqrt{1-y}$

$R = 2 + \sqrt{1-y}$.

answer=

$$V = \int_0^1 \pi (R^2 - r^2) \, dy.$$

$$= \pi \int_0^1 (2+\sqrt{1-y})^2 - (2-\sqrt{1-y})^2 dy.$$

9. (28 pts) Determine whether the series is convergent or divergent. Show your work and name the test(s) you are using.

(a) $\sum_{n=0}^{\infty} \frac{n}{n+1} = 1 + \sum_{n=1}^{\infty} \frac{n}{n+1}$

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

$\lim_{n\to\infty} \frac{\frac{n}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n+1} = \infty \Rightarrow \sum_{n=0}^{\infty} \frac{n}{n+1}$ diverges also

(b) $\sum_{n=0}^{\infty} \frac{2^n n^2}{n!}$ Ratio Test. $a_n = \frac{2^n n^2}{n!}$, $a_{n+1} = \frac{2^n 2 (n+1)^2}{(n+1) n!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2(n+1)}{n^2} = 0 < 1 \Rightarrow \sum_{n=0}^{\infty} \frac{2^n n^2}{n!}$ converges.

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ Integral Test

$u = \ln x$
$du = \frac{1}{x} dx$

$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}} dx = \lim_{a\to\infty} \int_2^a \frac{1}{x(\ln x)^{3/2}} dx = \lim_{a\to\infty} \left(\frac{-2}{\sqrt{\ln x}} \Big|_2^a \right)$

$= \lim_{a\to\infty} \left[\frac{-2}{\sqrt{\ln(a)}} + \frac{2}{\sqrt{\ln(2)}} \right) = \frac{2}{\sqrt{\ln(2)}} \Rightarrow$ integral converges $\Rightarrow$ series converges

(d) $\sum_{n=0}^{\infty} \frac{n!}{(n+2)!} = \sum_{n=0}^{\infty} \frac{n!}{(n+2)(n+1)n!} = \sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$

converges, LCT with $\sum \frac{1}{n^2}$.

Ratio test $a_n = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$

$\downarrow$ $a_{n+1} = \dfrac{(-3)^n(-3)\, x^n\, x}{\sqrt{n+2}}$

10. (7 pts) For the power series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ determine the open interval in which it converges. Do not bother to check the endpoints of the interval of convergence.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\cancel{|3|^n}|3|\,\cancel{|x|^n}|x|}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{|3|^n}\cancel{|x|^n}} = 3|x| < 1$$

$$|x| < \frac{1}{3}$$

The interval of convergence = $-\frac{1}{3} < x < \frac{1}{3}$

don't do $\rightarrow$ ~~11. (8 pts) The power series $\sum_{n=0}^{\infty} \frac{(x+2)^n}{n3^{n+1}}$ has radius of convergence 3. Determine whether the series is convergent or divergent at each endpoint of the interval of convergence.~~

Recall that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1.$$

right end point:__________

$\downarrow$

left end point:__________

12. (10 pts) Find the Maclaurin series $\sum_{n=0}^{\infty} a_n x^n$ for each of the following functions. Show your work. You are free to make use of the known expansion for $\frac{1}{1-x}$.

(a) $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

The series= __________

(b) $\tan^{-1} x$

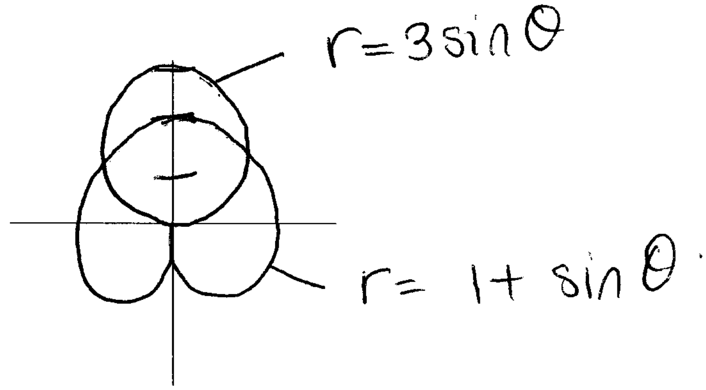
① $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

② $\int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$

The series=__________

③ $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ ✓

14. (12 pts) (a) On the same coordinate system, sketch the graphs of the equations $r = 3\sin\theta$ and $r = 1 + \sin\theta$.

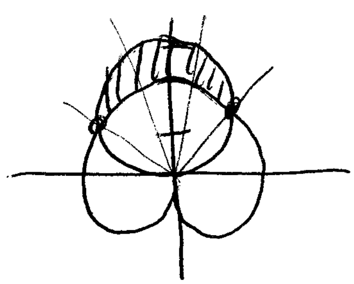


(b) Find all points of intersection of the graphs in (a).

$$3\sin\theta = 1 + \sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

The points of intersection are ______________________

(c) Set up (DO NOT EVALUATE) an integral for the area which is inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$



The integral is ______________________

End of Exam

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 - (1 + \sin\theta)^2\, d\theta.$$

$$f'(x) = -\sin(x) + x \quad f'(0) = 0$$
$$f''(x) = -\cos(x) + 1 \quad f''(0) = 0$$
$$f'''(x) = \sin(x) \quad f'''(0) = 0$$
$$f''''(x) = \cos(x) \quad f''''(0) = 1.$$
$$f^{v}(x) = -\sin(x)$$

$\boxed{a = 0}$

13. (16 pts) Consider the function $f(x) = \cos x - 1 + \frac{x^2}{2!}$.

p 808.

(a) Find the corresponding Taylor polynomial P_4.

$$P_4(x) = \frac{x^4}{4!}$$

$P_4 =$ ______________________

p 812

(b) Find the corresponding remainder R_4.

$$R_4(x) = \frac{f^{v}(c)(x)^5}{5!} = \frac{-\sin(c)x^5}{5!}$$

$R_4 =$ ______________________

(c) Find an estimate for R_4 which is valid for all x in the interval $(-1/10) \leq x \leq (1/10)$.

Since $|\sin(c)| \leq 1$ we have, whenever $|x| \leq \frac{1}{10}$

$$|R_4(x)| \leq \frac{1 \cdot |x|^5}{5!} \leq \frac{1}{120}\left(\frac{1}{10}\right)^5 \leq 10^{-7}$$

answer= .0000001

(Thus for x in this range, one can calculate $f(x)$ to 7 decimal places by simply finding $P_4(x) = x^4/24$.).