

# Convergence Tests — Summary

## Standard Series

1. **Geometric Series**  $\sum_{n=0}^{\infty} Ar^n = A + Ar + Ar^2 + \dots = \begin{cases} \frac{A}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$
2. **p-Series**  $\sum \frac{1}{n^p}$  converges if and only if  $p > 1$  (e.g.  $\sum \frac{1}{n^2}$  converges,  $\sum \frac{1}{\sqrt{n}}$  diverges).
3. **Constant Series**  $\sum^{\infty} c = c + c + c + \dots$  diverges (unless  $c = 0$ )
4. **Exponential Series**  $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$  (converges for any  $x$  by the ratio test).

## Our Tests

1. **Integral Test:** If  $f(x)$  is a continuous, non-negative, decreasing function, then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)dx \text{ is finite.}$$

2. **Comparison Test:** If  $0 \leq a_n \leq b_n$  for all large  $n$ , then  $\begin{cases} \sum b_n \text{ converges} \implies \sum a_n \text{ converges} \\ \sum a_n \text{ diverges} \implies \sum b_n \text{ diverges} \end{cases}$

3. **Limit Comparison Test:** If  $a_n, b_n \geq 0$  and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \quad \text{with } L \neq 0 \text{ or } \infty$$

then  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge.

*This makes precise the intuition that “ $a_n \approx Lb_n$  for large  $n$ ”. To apply it, take  $\sum b_n$  to be one of the “Standard Series” or one that can be handled with the integral test.*

4. **Ratio Test:** If  $a_n \geq 0$  and  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$  then  $\begin{cases} \text{if } r < 1 & \text{then } \sum a_n \text{ converges absolutely} \\ \text{if } r > 1 & \text{then } \sum a_n \text{ diverges} \\ \text{if } r = 1 & \text{can't tell} \end{cases}$

*This is useful for series involving exponentials (like  $2^n$ ) and factorials (like  $n!$ ).*

5. **Alternating Series Test:** If the  $a_n$  are non-negative ( $a_n \geq 0$ ), decreasing ( $a_1 \geq a_2 \geq a_3 \geq \dots$ ), and  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\sum (-1)^n a_n$  converges.

## Testing for Convergence

Check the convergence of a series  $\sum a_n$  by the following steps.

- (1) Check that  $\lim_{n \rightarrow \infty} |a_n| = 0$ . If not the series *diverges*.
- (2) Check  $\sum |a_n|$  by any test. If this converges  $\sum a_n$  is *absolutely convergent*.
- (3) If the series is alternating (i.e. of the form  $\sum (-1)^n |a_n|$ ) and the  $|a_n|$  are decreasing (for example, if derivative  $< 0$  for large  $x$ ) then the series is *conditionally convergent* by the A.S.T.

Note: If you apply the the ratio or root test to  $\sum |a_n|$  and get a limiting ratio  $r > 1$ , the series *diverges* and Step (3) is not needed.

For (2) ask yourself:

- Can I do the corresponding integral? If so, use the integral test.
- Can I simplify by dropping ‘lower order terms’? If so, justify this simplification by the L.C.T.
- Try the ratio test — especially if the terms involve factorials.
- Can I find an inequality comparing  $\sum a_n$  to a standard series? If so use the C.T.

## Power Series $\sum a_n(x - a)^n$

Always apply the Ratio Test to the series  $\sum |a_n(x - a)^n|$ . The condition  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x - a)^{n+1}}{a_n(x - a)^n} \right| < 1$  gives the inequality  $|x - a| < R$  for some  $R$  (possibly  $\infty$ ). The power series converges absolutely for each  $x$  inside the interval  $(a - R, a + R)$  and diverges for each  $x$  outside the interval.

(The values  $x = a \pm R$  on the boundary of this interval must be checked separately, but you won’t be asked to do this.)

## Remainders

1. **Integral Test Remainder Estimate:** Under the hypotheses of the Integral Test, the remainder

$R_N = S - S_N$  for the series  $\sum_{n=1}^{\infty} f(n)$  satisfies

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

2. **Alternating Series Remainder Bound:** Under the hypotheses of the Alternating Series Test,

the remainder  $R_N$  for the series  $\sum_{n=1}^{\infty} (-1)^n a_n$  satisfies

$$|R_N| \leq a_{N+1}.$$