

# Convergence Tests — Summary

## Standard Series

(a) **Geometric Series:**  $\sum_{n=0}^{\infty} Ar^n = A + Ar + Ar^2 + \dots = \begin{cases} \frac{A}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

(b) **p-Series:**  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$

(c) **Constant Series:** for any number  $c \neq 0$ ,  $\sum_{n=1}^{\infty} c = c + c + c + \dots$  diverges.

## Our Tests

0.  **$n^{\text{th}}$  Term Test:** If  $\lim_{n \rightarrow \infty} |a_n| \neq 0$  then  $\sum a_n$  diverges.

1. **Integral Test:** If  $f(x)$  is a continuous, non-negative, decreasing function, then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x) dx \text{ is finite.}$$

Use the next two tests to compare a given series to one of the “Standard Series” or one that can be handled with the integral test.

2. **Direct Comparison Test<sup>1</sup>:** If  $0 \leq a_n \leq b_n$  for all large  $n$ , then  $\begin{cases} \sum b_n \text{ converges} \implies \sum a_n \text{ converges} \\ \sum a_n \text{ diverges} \implies \sum b_n \text{ diverges} \end{cases}$

3. **Limit Comparison Test:** If  $a_n, b_n \geq 0$  and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$  with  $c \neq 0$  or  $\infty$ , then  $\sum a_n$  and  $\sum b_n$  do the same thing (both converge or both diverge). Furthermore,

- If  $c = 0$  then  $a_n \leq b_n$  for large  $n$ , and
- If  $c = \infty$  then  $b_n \leq a_n$  for large  $n$

so the Direct Comparison Test applies.

4. **Ratio Test:** If  $a_n \geq 0$  and  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$  then  $\begin{cases} \text{if } r < 1 & \text{then } \sum a_n \text{ converges absolutely} \\ \text{if } r > 1 & \text{then } \sum a_n \text{ diverges} \\ \text{if } r = 1 & \text{can't tell} \end{cases}$

Try this first for series where the “ $n$ ” appears in an exponent or as a factorial. Example:  $\sum \frac{2^n}{n!}$ .

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<sup>1</sup>In words: If a bigger series converges, then yours does too. If a smaller series diverges, then yours does too.