Series Tests — Complete Summary

Standard Series

1. **Geometric Series** $\displaystyle\sum_{n=0}^{\infty} Ar^n \;=\; A + Ar + Ar^2 + \cdots = \begin{cases} \frac{A}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

2. **p-Series** $\displaystyle\sum^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$ (e.g. $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{\sqrt{n}}$ diverges).

3. **Constant Series** $\sum^{\infty} c \;=\; c + c + c + \cdots$ diverges (unless $c = 0$)

4. **Exponential Series** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$ (converges for any x by the ratio test).

Our Tests

1. n^{th} **Term Test:** Is $\displaystyle\lim_{n\to\infty} |a_n| = 0$? If not, then $\sum a_n$ diverges.

2. **Integral Test:** If $f(x)$ is a continuous, non-negative, decreasing function, then

$$\sum_{n=1}^{\infty} f(n) \;\text{ converges} \iff \int_1^{\infty} f(x)dx \quad \text{is finite.}$$

3. **Comparison Test:** If $0 \leq a_n \leq b_n$ for all large n, then $\begin{cases} \sum b_n \text{ converges} \Longrightarrow \sum a_n \text{ converges} \\ \sum a_n \text{ diverges} \Longrightarrow \sum b_n \text{ diverges} \end{cases}$

4. **Limit Comparison Test:** If $a_n, b_n \geq 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \qquad \text{with } L \neq 0 \text{ or } \infty$$

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

This makes precise the intuition that "$a_n \approx Lb_n$ for large n". To apply it, take $\sum b_n$ to be one of the "Standard Series" or one that can be handled with the integral test.

5. **Ratio Test:** If $a_n \geq 0$ and $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ then $\begin{cases} \text{if } r < 1 & \text{then } \sum a_n \text{ converges absolutely} \\ \text{if } r > 1 & \text{then } \sum a_n \text{ diverges} \\ \text{if } r = 1 & \text{can't tell} \end{cases}$

This is useful for series involving exponentials (like 2^n) and factorials (like $n!$).

6. **Alternating Series Test:** If the a_n are non-negative ($a_n \geq 0$), decreasing ($a_1 \geq a_2 \geq a_3 \geq \cdots$), and $\lim_{n\to\infty} a_n = 0$, then $\sum (-1)^n a_n$ converges.

Testing for Convergence

Check the convergence of a series $\sum a_n$ by the following steps.

(1) Check that $\lim_{n\to\infty} |a_n| = 0$. If not the series *diverges.*

(2) Check $\sum |a_n|$ by any test. If this converges $\sum a_n$ is *absolutely convergent.*

(3) If the series is alternating (i.e. of the form $\sum(-1)^n|a_n|$) and the $|a_n|$ are decreasing (for example, if derivative < 0 for large x) then the series is *conditionally convergent* by the A.S.T.

Note: If you apply the the ratio or root test to $\sum |a_n|$ and get a limiting ratio $r > 1$, the series *diverges* and Step (3) is not needed.

For (2) ask yourself:

- Can I do the corresponding integral? If so, use the integral test.
- Can I simplify by dropping 'lower order terms'? If so, justify this simplification by the L.C.T.
- Try the ratio test — especially if the terms involve factorials.
- Can I find an inequality comparing $\sum a_n$ to a standard series? If so use the C.T.

Power Series $\sum a_n(x-a)^n$

Always apply the Ratio Test to the series $\sum |a_n(x-a)^n|$. The condition $\lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| < 1$ gives the inequality $|x-a| < R$ for some R (possibly ∞). The power series converges absolutely for each x inside the interval $(a-R, a+R)$ and diverges for each x outside the interval.

(The values $x = a \pm R$ on the boundary of this interval must be checked separately, but you won't be asked to do this.)

Taylor Series

Theorem If $f(x)$ has $n+1$ derivatives on an interval $[a, x]$ then

$$f(x) \;=\; f(a) \;+\; f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2 \;+\; \ldots + \frac{1}{n!}f^{(n)}(a)\,(x-a)^n \;+\; R_n$$

where the remainder satisfies

$$|R_n| \;\leq\; M\,\frac{|x-a|^{n+1}}{(n+1)!} \qquad \text{where} \quad M = \text{max. of } |f^{(n+1)}(t)| \text{ for } t \text{ between } a \text{ and } x.$$

Important Taylor Series

These series converge for any x:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

and these converge on the interval $|x| < 1$:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \\
-\ln(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \\
(1+x)^p &= 1 + px + \frac{p(p-1)}{2!}\,x^2 + \frac{p(p-1)(p-2)}{3!}\,x^3 + \cdots = 1 + \sum_{n=1}^{\infty} \binom{p}{n} x^n.
\end{aligned}$$