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A MORSE THEORY FOR EQUIVARIANT YANG-MILLS

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0. Introduction. This paper develops a Morse theory for equivariant Yang-Mills and Yang-Mills-Higgs fields on compact Riemannian 4-manifolds. One immediate consequence is the first proof of the existence of nontrivial Yang-Mills-Higgs fields on S^4. Another is a topological criteria for the existence of non-self-dual Yang-Mills fields: for appropriate Lie group actions, if the space of equivariant connections does not retract to the equivariant moduli space, then there exist Yang-Mills connections that are neither SD nor ASD (Theorem 3.4).

The Yang-Mills equations arise as a variational problem. Let $P \to M$ be a principal G-bundle over a compact Riemannian manifold. Each connection A on P has a curvature 2-form F^A. The Yang-Mills action

$$YM(A) = \int_M |F^A|^2 \, dv \tag{0.1}$$

is a function on the space $\mathscr{A}$ of connections whose critical points are the YM fields. More generally, we can (following the physicists) choose a hermitian vector bundle E associated to P and define, for each connection A on P and each section ϕ of E, the Yang-Mills-Higgs action

$$YMH(A, \phi) = \frac{1}{2} \int_M |F^A|^2 + |d^A\phi|^2 + \frac{1}{2}(|\phi|^2 - \mu)^2 \, dv \tag{0.2}$$

where $\mu > 0$ is a constant, the "mass parameter". The critical points are Yang-Mills-Higgs (YMH) fields. (The last term, the "Higgs potential", is included with the aim of ensuring that the Lagrangian has nontrivial minima; without it, YMH would be minimized by $\phi \equiv 0$.) Both (0.1) and (0.2) are invariant under the gauge group $\mathscr{G}$ and hence descend to functions on the orbit spaces $\mathscr{B} = \mathscr{A}/\mathscr{G}$ and $\mathscr{E} = \mathscr{A} \times_{\mathscr{G}} \Gamma(E)$.

One can view this situation from the perspective of Morse theory. When M is 4-dimensional, the Yang-Mills function on $\mathscr{B}$ is minimized along the moduli space $\mathscr{M}$ of self-dual/anti-self-dual connections. Other, nonminimal critical points have recently been discovered ([SSU], [SS], [Pk3]). A Morse theory for YM could provide a conceptually simple method for obtaining such nonminimal critical YM fields. A Morse theory for the YMH action would be even more useful because no nontrivial YMH fields are known. Unfortunately, the usual formulations of Morse

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theory in infinite dimensions require a compactness condition, Condition C, that fails for YM and YMH in dimension 4 (although it holds in dimensions ≤ 3). The aim of this paper is to show that one can recover Condition C and obtain a good Morse theory by imposing symmetry.

The key ingredient is the equivariant Sobolev theorem proved in Section 1. Suppose that a compact Lie group H acts isometrically on a hermitian vector bundle $E \to M$. By completing the set $\Gamma_H(E)$ of equivariant sections in appropriate norms, we obtain Sobolev spaces of equivariant sections. The equivariant Sobolev theorem shows that, when all orbits have positive dimension, these spaces satisfy Sobolev inequalities as if they were functions on a lower-dimensional manifold (even when the orbit space is not a manifold). This gives a direct way to approach variational PDEs that are borderline for the Sobolev embeddings: look for solutions invariant under an isometric group action with all orbits of dimension $d > 1$, and use the equivariant Sobolev theorem to verify Condition C on the space of invariant functions. By the work of R. Palais ([P1], [P2]), one can then apply Lusternik-Schnirelman theory and (provided that all critical points are nondegenerate) Morse theory. This yields points that are critical with respect to *equivariant* perturbations. Under quite general circumstances this implies criticality with respect to all variations (this is the "symmetric criticality principle" [P4]). Elliptic theory then shows that these critical points are smooth solutions of the variational equations.

When applicable, this procedure quickly reduces PDE existence theorems to equivariant homotopy theory (e.g., finding the Lusternik-Schnirelman category of the space of invariant functions). It can be applied to each of the important geometric problems that involve borderline Sobolev embeddings, such as the Yamabe and constant mean curvature equations.

When this approach is applied to the Yang-Mills equations, there are complications caused by the gauge group. An H-action on the principal bundle P defines a space $\mathscr{A}^H$ of invariant connections that is preserved by a subgroup $\mathscr{G}^H \subset \mathscr{G}$ of the gauge group. The functions (0.1) and (0.2) then descend to functions on $\mathscr{B}^H = \mathscr{A}^H/\mathscr{G}^H$ and $\mathscr{E}^H = \mathscr{A}^H \times_{\mathscr{G}^H} \Gamma_H(E)$ respectively. In Section 3 we show that, under appropriate conditions on the H-action, one can apply Morse theory on these spaces to deduce the existence of YM and YMH fields. (The needed analysis is developed in Section 2 and the appendix.) For applications we turn to the quadrapole bundles introduced in [ASSS]. Section 4 gives a self-contained description of the quadrapole bundles and uses the G-index theorem to show that most admit no SD Yang-Mills fields. These bundles are then used in Section 5 to establish the existence of nontrivial Yang-Mills-Higgs fields on S^4. Along the way, we obtain a new proof of the result of Sadun and Segert on the existence of nonminimal Yang-Mills fields on S^4.

The equivariant Morse theory shows that topology forces the existence of at least some critical points of the YM and YMH functions. One should bear in mind, however, that the critical point structure of these functions depends on the geometry (not just the topology) of the base manifold M. In [Pk3] it is shown that there exist unstable Yang-Mills on (S^4, g) for a family of equivariant metrics g that includes metrics arbitrarily close to the standard metic g_0. These Yang-Mills fields have no limit as $g \to g_0$ and represent critical points not forced by topology.

1. The equivariant Sobolev theorem. Let (M, g) be a compact n-dimensional Riemannian manifold and let E be a vector bundle over M with a fixed fiber metric and compatible connection ∇. As usual, the Sobolev space $L^{k,p}(E)$ is the completion of $\Gamma(E)$ with respect to the norm

$$\|\phi\|_{k,p} = \left(\sum_{l=0}^{k} \int_M |\nabla^l \phi|^p\right)^{1/p} \tag{1.1a}$$

where $\nabla^l \phi = \nabla \circ \nabla \circ \cdots \circ \nabla \phi \in \Gamma(\bigotimes^l T^*M \otimes E)$ is the lth covariant derivative of ϕ. The Holder spaces $C^{k,\alpha}(E)$ are defined similarly using the norm

$$\|\phi\|_{k,\alpha} = \sum_{l=0}^{k} \sup|\nabla^l \phi| + \sup_{x,y} \frac{|\nabla^k \phi(x) - \nabla^k \phi(y)|}{|\mathrm{dist}(x, y)|^\alpha} \tag{1.1b}$$

where the last supremum is over all $y \neq x$ contained in a normal coordinate neighborhood of x and $\nabla^l \phi(y)$ is taken to mean the tensor at x obtained by parallel transport along the geodesic from x to y. The following fundamental result is well known.

SOBOLEV EMBEDDING THEOREM. *Let $1 \leqslant p < q$, $k \geqslant l$, and $0 < \alpha < 1$. Then*

(i) *for $k - n/p \geqslant l - n/q$ the identity map induces a continuous inclusion $L^{k,p}(E) \hookrightarrow L^{l,q}(E)$, and this inclusion is compact if $k > l$ and $k - n/p > l - n/q$, and*

(ii) *for $k - n/p \geqslant l + \alpha$ the inclusion $L^{k,p}(E) \hookrightarrow C^{l,\alpha}(E)$ is continuous, and it is compact if $k - n/p > l + \alpha$.*

Next, suppose that a compact Lie group H acts isometrically on E, that is, each $h \in H$ gives a smooth metric-preserving bundle map $\tilde{h}: E \to E$ covering an isometry $h: M \to M$. A section $\phi \in \Gamma(E)$ is equivariant if $\phi(h(x)) = \tilde{h}(\phi(x))$ for all $x \in M$. Let $\Gamma_H(E)$ denote the set of all smooth equivariant sections. (A word on terminology: these sections are *equivariant* for the action on E and are *invariant* for the induced action on $\Gamma(E)$; we will use these words interchangably.)

The group also acts on the connections on E, with $h \in H$ taking ∇ to the connection $h \cdot \nabla$ defined by

$$(h \cdot \nabla)_X \phi = \tilde{h}^{-1}(\nabla_{h_* X}(\tilde{h}\phi)), \qquad \forall X \in \Gamma(TM), \qquad \phi \in \Gamma(E). \tag{1.2}$$

By the devise of averaging over the group, we obtain an invariant connection ∇^0, which we fix once and for all. Completing the set $\Gamma_H(E)$ with respect to the norms (1.1)—using the invariant connection ∇^0—we obtain Sobolev spaces $L^{k,p}_H(E)$

and $C_H^k(E)$. These are closed subspaces of $L^{k,p}(E)$ and $C^k(E)$; so the Sobolev embeddings of Theorem 1 restrict to corresponding embeddings for $L_H^{k,p}(E)$ and $C_H^k(E)$. However, when the H-orbits have positive dimension, one expects a better result: since equivariant sections correspond, in some sense, to sections over the orbit space M/H, the spaces $L_H^{k,p}(E)$ should satisfy the Sobolev theorem with the dimension n replaced by $\dim(M/H)$. While this is not correct, the following result shows that equivariance does indeed yield improved Sobolev embeddings.

EQUIVARIANT SOBOLEV EMBEDDING THEOREM. *Suppose that a compact Lie group H acts isometrically on E and that each H-orbit in M has dimension $\geqslant d$. Then the spaces $L_H^{k,p}(E)$ and $C_H^k(E)$ satisfy the Sobolev embedding theorem with n replaced by $n - d$.*

Proof. By the differentiable slice theorem, the orbit through each $x \in M$ has a tubular neighborhood $\mathscr{U}_x$ equivariantly diffeomorphic (by the exponential map) to $H \times_{H_x} B_x$, where H_x is the isotropy subgroup at x and B_x is the ball of radius ε at the origin in the normal space to the orbit at x. By making these balls B_x smaller if necessary, we may assume that the metric on $\mathscr{U}_x$ is uniformly close to the product metric on $H \times_{H_x} B_x$. Let $\{\mathscr{U}_i\}$, $1 \leqslant i \leqslant l$, be a finite subcover of $\{\mathscr{U}_x / x \in M\}$. Write E_i for the restriction of the bundle E to $\mathscr{U}_i$, set

$$L_H^{k,p}(\{E_i\}) = \{(\phi_1, \dots, \phi_l) \in \bigoplus L_H^{k,p}(E_i) \colon \phi_i|_{\mathscr{U}_j} = \phi_j|_{\mathscr{U}_i} \; \forall i, j\}, \tag{1.3}$$

and define $C_H^k(\{E_i\})$ similarly. We then have isomorphisms

$$L_H^{k,p}(E) \simeq L_H^{k,p}(\{E_i\}), \qquad C_H^k(E) \simeq C_H^k(\{E_i\}) \tag{1.4}$$

(see Theorem 4.3 [P3]). Furthermore, each equivariant section ϕ on $\mathscr{U}_x$ pulls back by the exponential map to an equivariant section $\hat{\phi}$ on $H \times_{H_x} B_x$, and the $L^{k,p}$ (resp. C^k) norm of ϕ on $\mathscr{U}_x$ is uniformly equivalent to the $L^{k,p}$ (resp. C^k) norm of ϕ^* on $H \times_{H_x} B_x$. Together with (1.3) and (1.4), this means that it suffices to prove the theorem for equivariant Sobolev spaces on $H \times_{H_x} B_x$ with its product metric.

Now consider the restriction map $\Gamma_H(H \times_{H_x} S) \to \Gamma_H(B)$, where S is the slice $\mathrm{Id} \times B_x$. Each equivariant section ϕ satisfies

$$|\phi(h(x))| = |\tilde{h}\phi(x)| = |\phi(x)|; \tag{1.5}$$

i.e., $|\phi|$ is constant on orbits. Likewise, (1.2) and the invariance of ∇^0 imply that $|(\nabla^0_{h_* X}\phi)(h(x))| = |(\nabla^0_X \phi)(x)|$. This total covariant derivative $\nabla^0\phi$ splits into components tangent to the orbits and those tangent to the slice.To do this explicitly, we decompose the Lie algebra of H as $\mathfrak{h} = \mathfrak{h}_x \oplus \mathfrak{m}$, where $\mathfrak{h}_x$ is the Lie algebra of the isotropy subgroup and $\mathfrak{m}$ is its orthogonal complement. The tangent space $T(H \times_{H_x} B_x)$ at each point (Id, y) on the slice is then isometrically identified with $\mathfrak{m} \oplus T_y B$. Thus, if we choose an orthonormal basis $\{e_i\}$ of $\mathfrak{m}$, the corresponding vector fields on $H \times_{H_x} B_x$ are perpendicular to the slice, and we have

$$|\nabla^0\phi|^2 = |(\nabla^0\phi)^T|^2 + \sum|\nabla^0_{e_i}\phi|^2 \tag{1.6}$$

where T denotes the component tangent to S. The last term in this expression is in fact algebraic in ϕ, as follows.

Each $X \in \underline{h}$ gives a vector field $\tilde{X}$ on E. Because the flow of X preserves ϕ, the Lie derivative $\mathscr{L}_{\tilde{X}}\phi$ vanishes. On the other hand, the difference between the covariant derivative and the Lie derivative is a zeroth order operator. Thus, $\Psi_X = \nabla^0_X - \mathscr{L}_{\tilde{X}}$ is a smooth endomorphism of E; it and its covariant derivatives depend only on ∇^0 and the (lifted) H-action. Therefore, (1.6) can be written as

$$|\nabla^0\phi|^2 = |\nabla^0(\bar{\phi})|^2 + \sum|\Psi_{e_i}(\bar{\phi})|^2 \tag{1.7}$$

where $\bar{\phi}$ is the restriction of ϕ to the slice. From (1.5), (1.7), and the fact that Ψ is bounded, we see that the C^1_H-norm of ϕ on $H \times_{H_x} B_x$ is equivalent to the C^1 norm of $\bar{\phi}$ on the slice S. Likewise, since

$$\int_{H \times_{H_x} B_x} |\nabla^0\phi|^p = \mathrm{Vol}(H/H_x)\int_B |\nabla^0\phi|^p,$$

the $L^{1,p}_h$-norm of ϕ on $H \times_{H_x} B_x$ is equivalent to the $L^{1,p}$-norm of $\bar{\phi}$ on S. Similar statements hold higher derivatives, giving isomorphisms

$$L^{k,p}_H(E) \simeq L^{k,p}_{H_x}(E|_{\text{slice}}), \qquad C^k_H(E) \simeq C^k_{H_x}(E|_{\text{slice}}).$$

Finally, since each H-orbit has dimension $\geqslant d$, each slice $B_i = \mathrm{Id} \times B_{x_i}$ has dimension $\leqslant n - d$; so the $(n - d)$-dimensional Sobolev embeddings hold on each slice. ■

The equivariant Sobolev theorem is applicable to a variety of geometric PDE problems. In the subsequent sections we will focus on one of particular interest: the existence of Yang-Mills and Yang-Mills-Higgs fields.

2. The YM and YMH functions. A standard calculation shows that the variational equations of the YM function (0.1) are the Yang-Mills equations $(d^A)^*F^A = 0$. The variational equations for the YMH (0.2) function are computed similarly (cf. [Pk1]). The result is a pair of coupled nonlinear PDEs in the variables (A, ϕ), the Yang-Mills-Higgs equations

$$\begin{cases} (d^A)^*F + \mathrm{Re}\langle d^A\phi, \rho(\ )\phi\rangle = 0 & \text{(2.1a)} \\ (d^A)^*d^A\phi + (|\phi|^2 - \mu)\phi = 0. & \text{(2.1b)} \end{cases}$$

The first equation involves the $(Ad\ P)^*$-valued 1-form given by

$$\mathrm{Re}\langle d^A\phi, \rho(\ )\phi\rangle(X\otimes\sigma) = \mathrm{Re}\langle d^A_X\phi, \rho(\sigma)\phi\rangle \qquad \forall X\otimes\sigma\in\Gamma(TM\otimes Ad\ P).$$

Both the Lagrangian and the equations are gauge invariant and conformally invariant.

In several cases the equations decouple, giving rather trivial solutions (A, ϕ):

(i) A is a Yang-Mills field, and $\phi\equiv 0$.

(ii) A is the trivial connection on the trivial bundle, and $\phi\equiv\sqrt{\mu}$ is the constant function.

(iii) A is a Yang-Mills field, E is associated to P by a representation ρ that contains a trivial representation (so $E = E'\oplus\tau$, where τ is a trivial line bundle), and $\phi\in\Gamma(\tau)$ satisfies $d^*d\phi + (|\phi|^2-\mu)\phi = 0$.

We will refer to solutions of types (i)–(iii) as "decoupled solutions".

To date, no coupled solutions are known. This is in sharp contrast to the Yang-Mills equations, where one has Taubes's existence theorems for self-dual solutions. The next result helps explain why the YMH equations are more difficult than the YM equations.

PROPOSITION 2.1. *For $G = SU(2)$ a stable Yang-Mills-Higgs field (A, ϕ) on S^4 is decoupled and A is SD or ASD and ϕ is constant.*

Proof. There is a standard method, originally due to J. Simons, for checking the stability properties of conformally invariant equations on spheres. The unit sphere $S^n\subset\mathbb{R}^{n+1}$ has $n+1$ conformal vector fields $\{X_i\}$ obtained by projecting the basis vector fields $\partial/\partial x^i$ onto the sphere. One computes the Hessian of the Lagrangian, evaluates on each X_i, and sums on i. For a YMH field (A, ϕ) on S^4, one finds ([Pk2, Cor. 3.5])

$$\sum \mathrm{Hess}(YMH)(X_i, X_i) = -4\int_{S^4}|d^A\phi|^2.$$

Thus, stability implies that $d\phi = 0$. Then by (2.1), $\phi\equiv 0$ or $|\phi|^2\equiv\mu$, and A is Yang-Mills. In fact, A is a stable Yang-Mills connection (by comparison with nearby fields (A, ϕ) with ϕ fixed). For $G = SU(2)$ every stable YM field on S^4 is SD or ASD ([BL]). Moreover, if $|\phi|^2\equiv\mu\neq 0$, then ϕ is a parallel nowhere-vanishing section of E, and so splits off a trivial line bundle τ from E. Then either E is associated to P by a representation ρ that contains a trivial representation or the connection is reducible. But every reducible Yang-Mills connection on the trivial bundle on S^4 is gauge equivalent to the trivial connection ([FU]). ■

Proposition 2.1 implies that on S^4 there is no first-order system, analogous to the SD/ASD equations, that gives absolute minima of the Lagrangian.[1] From this

[1] When M is a Kahler manifold, there is a version of the YMH Lagrangian that is minimized by the solutions of the first-order "Bogomolny equations".

viewpoint the existence of YMH fields is analogous to the existence of *nonminimal* Yang-Mills fields.

The simplest way to find nonminimal critical points is to construct a Morse theory or Lusternik-Schnirelman theory. Much of the basic framework for Morse theory for the YM and YMH functions already exists in the literature ([U], [T1], [T2], [Pk1]). In the remainder of this section we quickly review this, presenting it in a way that is useful for the next section and explaining why a true Morse theory does not exist.

As in the introduction, we start with a compact oriented Riemannian 4-manifold (M, g), a principal bundle $P \to M$ with compact structure group G, and an associated vector bundle $E \to M$. Let $\mathscr{A}$ denote the space of connections on P and let $\mathscr{G} = \Gamma(P \times_{\mathrm{Ad}} G)$ be the gauge group.

These spaces have Sobolev completions $\mathscr{A}^{k,p}$, $\mathscr{G}^{k,p}$ constructed in the standard manner ([FU]). We will fix somewhat unconventional norms and will omit the superscripts. Specifically, let $\mathscr{A}$ (resp. $L(E)$) be the completion of the space of connections (resp. $\Gamma(E)$) in the $L^{1,2} \cap L^5$ norm and let $\mathscr{G}$ be the completion of the gauge group in the $L^{2,2} \cap L^{1,5}$ norm. As usual, $\mathscr{A}^* \subset \mathscr{A}$ will denote the set of irreducible connections. With these norms we have the following lemma.

LEMMA 2.2. *$\mathscr{G}$ is a smooth Lie group, the action of $\mathscr{G}$ on $\mathscr{A}$ and $L(E)$ is smooth, and the YM (resp. YMH) function is a smooth $\mathscr{G}$-invariant function on $\mathscr{A}$ (resp. $\mathscr{A} \times L(E)$). Moreover, the orbit spaces $\mathscr{B}^* = \mathscr{A}^*/\mathscr{G}$ and $\mathscr{E}^* = \mathscr{A}^* \times_{\mathscr{G}} L(E)$ are smooth manifolds, $\mathscr{E}^* \to \mathscr{B}^*$ is a smooth vector bundle, and YM and YMH descend to smooth functions*

$$YM\colon \mathscr{B}^* \to \mathbb{R}, \qquad YMH\colon \mathscr{E}^* \to \mathbb{R}. \tag{2.2}$$

Proof. The 4-dimensional Sobolev embedding $L^{1,5} \subset C^0$ means that the elements of $\mathscr{G}$ are continuous, and therefore $L^{2,2} \cap L^{1,5}$ is a Banach algebra. The statements of the first sentence follow exactly as in Section 1 of [U] and Section 4 of [Pk1]. The remaining statements follow from a slice theorem as in Section 3 of [FU] and Section 4 of [Pk1]. ∎

We can also use the Sobolev norm to make $\mathscr{B}^*$ and $\mathscr{E}^*$ into *Riemannian* manifolds. At each $\nabla \in \mathscr{A}$ there is a natural identification $T_\nabla \mathscr{A} = L^{1,2} \cap L^5(T^*M \otimes \mathrm{Ad}(P))$, where $\mathrm{Ad}(P)$ is the adjoint bundle of P. One can then define an inner product on $T_\nabla \mathscr{A}$ by

$$\langle X, Y \rangle_A = \int_M \langle \nabla^A X, \nabla^A Y \rangle + \langle X, Y \rangle. \tag{2.3}$$

This defines a smooth $\mathscr{G}$-invariant Riemannian metric on $\mathscr{A}$, and there is a similar $L^{1,2}$ metric on $\mathscr{A} \times L(E)$. (These metrics are not complete since $\mathscr{A}$ is defined using a norm stronger that $L^{1,2}$.) These metrics descend, defining smooth Riemannian metrics on $\mathscr{B}^*$ and $\mathscr{E}^*$ by the requirement that $\mathscr{A}^* \to \mathscr{B}^*$ and $\mathscr{A}^* \times L(E) \to \mathscr{E}^*$ be

Riemannian submersions. We can then construct the gradient vector fields of the functions (2.2).

For Morse theory in infinite dimensions, one also needs a compactness condition, Condition C. Suppose f is a function on a Riemannian manifold X that is bounded below. A sequence $\{x_i\} \in X$ is *Palais-Smale* if $\{|f(x_i)|\}$ is bounded and if $\{|(\text{grad } f)_{x_i}|\} \to 0$ as $i \to \infty$. The function f satisfies Condition C if every Palais-Smale sequence in X has a convergent subsequence. This insures that the downward gradient flow lines of f converge. When it holds, the usual results of Morse and Lusternik-Schnirelman theory follow ([P1], [P2]).

Condition C fails for the YM and YMH functions and fails in a very specific way. The following result is proved in the appendix; it is an extension of a result of Uhlenbeck, Sedlacek, and Taubes.

THEOREM 2.3. *Let $\{\nabla^i, \phi^i\} \in \mathscr{E}$ be an Palais-Smale sequence for the Yang-Mills-Higgs function on a bundle $P \to M$ (using the $L^{1,2}$ norm on $\mathscr{E}$). Then there is a finite set of points $\{x_l\}$, consisting of exactly those points $x \in M$ with*

$$\lim_{\varepsilon \to 0} \limsup_{i \to \infty} \int_{B(x,\varepsilon)} |F^i|^2 + |\nabla\phi|^2 + |\phi|^4 \neq 0 \tag{2.4}$$

and gauge transformations $\{g^i\} \in \mathscr{G}$ such that a subsequence of the $\{g^i \cdot \nabla^i\}$ converges strongly in $L^{1,2}_{\text{loc}}$ on $M - \{x_l\}$ to a smooth critical point (∇, ϕ) on a bundle $P' \to M$.

This falls short of Condition C in several respects. First, the convergence in the conclusion is too weak: $\mathscr{E}$ is complete in the $L^{1,2} \cap L^5$ norm, but sequences that converge in $L^{1,2}$ may not have a limit. Second, convergence fails at a finite set of points, and the limit connection can be on a topologically different bundle—this is the well-known "bubbling" phenomenon. There is also the problem that the $L^{1,2}$ Riemannian metric on $\mathscr{B}^*$ is not complete. We will see next that all three of these difficulties are avoided under an equivariance assumption.

3. Equivariant connections. We now return to the situation of Section 1 and consider a compact Lie group H acting smoothly and isometrically on M. Suppose that this action preserves orientation and that $h^*P \cong P$ for all $h \in H$. Then for each $h \in H$ there is a bundle automorphism γ_h: $P \to P$ covering h. This γ_h is unique up to a gauge transformation; so there is a well-defined action of H on $\mathscr{B}$ given by $[\nabla] \mapsto [\gamma_h \cdot \nabla]$. Let $\mathscr{B}^H$ denote the fixed set of this action.

There is an alternative way of describing the action of H on $\mathscr{B}$. Let $Aut(P)$ be the space of all bundle automorphisms of P. There is an obvious projection π from $Aut(P)$ to the diffeomorphism group on M and an exact sequence

$$\{1\} \to \mathscr{G} \to Aut(P) \xrightarrow{\pi} Diff(M). \tag{3.1}$$

A *lift* of the H-action is a homomorphism $H \to Aut(P)$ covering the action of H on M. (In Section 1 we assumed a lift as given.) Each lift gives an action of H on $\mathscr{A}$

and hence an action on $\mathscr{B}$ that clearly agrees with the one defined above. The relation between these two descriptions has been clarified by Fintushel-Stern ([FS]) and Braam-Matic ([BM]) and goes as follows.

Let $Aut_H \subset Aut(P)$ be the preimage of $H \subset Diff(M)$ under π. Given a fixed point $[A] \in \mathscr{B}^H$, choose a representative $A \in \mathscr{A}$ with covariant derivative ∇^A. Then the action of Aut_H on $\mathscr{A}$ has a stabilizer $\mathscr{H}_A$, and there is an exact sequence of finite-dimensional Lie groups

$$\{1\} \to \mathscr{G}_A \to \mathscr{H}_A \xrightarrow{\pi} H \to \{1\} \tag{3.2}$$

where $\mathscr{G}_A = \ker \pi = \{\gamma \in \mathscr{G} | \nabla^A \gamma = 0\}$. A lift of the action is exactly a splitting of this sequence, and we say that two such lifts are *equivalent* if they differ by conjugation by an element of $\mathscr{G}$. Since $\mathscr{H}_{\gamma \cdot A} = \gamma \mathscr{H}_A \gamma^{-1}$, the equivalence class of the lift is independent of the representative ∇ of the gauge orbit.

Let I be a set parameterizing equivalence classes of H-lifts. For each $i \in I$ let $\mathscr{A}_i^* \subset \mathscr{A}^*$ be the set of irreducible connections fixed by the H-action labeled by i, and $\mathscr{G}_i$ the subgroup of gauge transformations that commute with that lift. If $[A] \in \mathscr{B}^H$ is irreducible, then $\mathscr{G}_A = \mathrm{Id}$; so there exists a unique lift by (3.2). Furthermore, if two connections $A, A' \in \mathscr{A}_i^*$ invariant under this lift are related by $A' = g \cdot A$ for some $g \in \mathscr{G}_i$, then the action $h_i\colon P \to P$ of each $h \in H$ satisfies $h_i g \nabla = h_i \nabla' = \nabla' = g\nabla = g h_i \nabla$; so $[h_i, g](\nabla) = \nabla$. But then $\nabla([h_i, g]) = 0$; so $[h_i, g] \in \mathscr{G}_A = \mathrm{Id}$, which means that $g \in \mathscr{G}_i$. Thus, the irreducible part of $\mathscr{B}^H$ is a disjoint union

$$(\mathscr{B}^*)^H = \bigcup_{i \in I} (\mathscr{A}_i^*/\mathscr{G}_i) = \bigcup_{i \in I} \mathscr{B}_i^*. \tag{3.3}$$

Letting $\mathscr{E}_i$ denote the restriction of $\mathscr{E} \to \mathscr{B}$ to $\mathscr{B}_i^*$ yields an analogous decomposition

$$(\mathscr{E}^*)^H = \bigcup_{i \in I} (\mathscr{E}_i^*/\mathscr{G}_i) = \bigcup_{i \in I} \mathscr{E}_i^*. \tag{3.4}$$

The following theorem constructs our Morse theory for equivariant connections.

THEOREM 3.1. *Suppose that a compact Lie group H acts smoothly and isometrically on M as orientation preserving diffeomorphisms. Suppose also that* (i) $h^*P \cong P$ $\forall h \in H$, (ii) *all H-orbits have dimension at least one, and* (iii) $\mathscr{B}_i^H$ *contains no reducible connections. Then*

(a) *$\mathscr{B}_i$ (resp. $\mathscr{E}_i$) is a smooth closed submanifold of $\mathscr{B}$ (resp. $\mathscr{E}$) and is a complete Riemannian manifold with respect to the $L^{1,2}$ metric* (2.3),

(b) *on $\mathscr{B}_i^H$ (resp. $\mathscr{E}_i^H$) the Yang-Mills (resp. Yang-Mills-Higgs) are smooth functions that satisfy Condition C and hence satisfy Morse and Lusternik-Schnirelman theory, and*

(c) *the critical points are smooth YM (resp. YMH) fields.*

Proof. Since reducible connections in $\mathscr{B} - \mathscr{B}_i^H$ are isolated, we can restrict attention to a neighborhood of $\mathscr{B}^H$, and then a general result of Palais, Lemma 3.2

below, gives (a), except for the completeness assertion. Completeness follows provided each $\mathscr{A}_i$ is a closed submanifold of $\mathscr{A}$ (with its $L^{1,2} \cap L^5$ topology). Fix a connection $\nabla^i \in \mathscr{A}_i$. Then the correspondence $\nabla \mapsto A = \nabla - \nabla^i$ is a bounded linear isomorphism between $\mathscr{A}_i$ and the space

$$L_H^{1,2}(T^*M \otimes Ad\, P)$$

of H-invariant $Ad\, P$-valued 1-forms. But $L_H^{1,2} \hookrightarrow L^{1,2} \cap L^5$ is bounded by the equivariant Sobolev theorem. Thus, $\mathscr{A}_i \subset \mathscr{A}$ is a closed linear subspace. The argument for $\mathscr{E}_i^H$ is identical.

Condition C follows easily from Theorem 2.3. For equivariant pairs $(A, \phi) \in \mathscr{E}^H$, the integrand of (4.1) is an invariant function. Hence, $\{x_l\}$ is an invariant set, which must be empty since there are no zero-dimensional orbits. Furthermore, the equivariant Sobolev embedding $L_H^{1,2} \hookrightarrow L^5$ shows that $L^{1,2}$ convergence of equivariant connections implies convergence in $\mathscr{E}$. Since $\mathscr{E}_i^H$ is closed, the limit lies in $\mathscr{E}_i^H$.

We can therefore apply the general constructions of Morse and Lusternik-Schnirelman theory ([P1], [P2]) on $\mathscr{E}^H$ to obtain points critical with respect to equivariant variations. But Lemma 3.2 also says that the symmetric criticality principal holds; so these are critical points with respect to all variations. Smoothness then follows by elliptic regularity. ■

LEMMA 3.2 ([P4], §4–5). *Let H be a compact Lie group acting smoothly on a smooth Banach manifold X. If $f: X \to \mathbb{R}$ is a smooth H-invariant function, then the set $X^H \subset X$ of fixed points is a smooth closed invariant submanifold and the symmetric criticality principle holds.* ■

Later, applying Theorem 3.1, we will use the following simple criterion.

LEMMA 3.3. *When $G = SU(2)$, the assumptions of Theorem 3.1 hold provided H is connected, $-c_2(P)$ is not the square of an element in $H^2(M; \mathbb{Q})^H$, and all H-orbits have dimension at least one.*

Proof. Since H is connected, each $h \in H$ preserves the orientation and the homotopy class of the classifying map of P; so $h^*P \cong P$. An equivariant reducible $SU(2)$ connection reduces P to an S^1 bundle Q whose first Chern class $c_1(Q)$ lies in $H^2(M; \mathbb{Q})^H$ and satisfies $c_1(Q) \cdot c_1(Q) = -c_2(P)[M]$. □

The Morse Theorem 3.1 reduces the existence of nonminimal YM fields to a homotopy problem. (It has similar implications for YMH fields; see Section 5 below.)

THEOREM 3.4. *Under the conditions of Theorem 3.1, if the inclusion*

$$\mathscr{M}^H \hookrightarrow \mathscr{B}^H$$

of the invariant self-dual moduli space into $\mathscr{B}^H$ is not a homotopy equivalence, then there are nonminimal Yang-Mills fields on P. □

The moduli space $\mathcal{M}^H$ is finite-dimensional; so if one could show that $\mathcal{B}^H$ has infinite cohomological dimension, it would force the existence of nonminimal Yang-Mills fields. (In fact, dim $\mathcal{M}^H$ is at most the dimension of $\mathcal{M}$ given by the standard index formula; so one only needs a nonzero cohomology class in high dimension.) It would be interesting to have a practical method of computing $H^*(\mathcal{B}^H)$ and to carry through this argument in some specific examples.

In subsequent sections we will apply this general result to $SU(2)$ actions on the 4-sphere. For this we need specific equivariant bundles over S^4.

4. Quadrapole bundles. Quadrapole bundles are a family of $SU(2)$-equivariant quaternionic line bundles over S^4 that originally arose in quantum mechanics [ASSS]. Sadun and Segert [SS] have used explicit ODE computations to show that most of these bundles admit no SD or ASD Yang-Mills fields. Here we give a topological proof of that result.

First, recall the structure of the representation ring of $SU(2)$. The irreducible representations of $SU(2)$ are the symmetric powers $D_{l/2} = \mathrm{Sym}^l\, \mathbb{C}^2$ of the usual representation on $\mathbb{C}^2$ and all representations are unitary. The dimension of $D_{l/2}$ is $l + 1$; for l even it is orthogonal and real (i.e., it is the complexification of a real representation $D^r_{l/2}$), and for l odd it is symplectic and quaternionic. Tensor products decompose according to the Clebsch-Gordan formula

$$D_{k/2} \otimes D_{l/2} = D_{k+l/2} \oplus D_{(k+l-2)/2} \oplus \cdots \oplus D_{|k-l|/2}\,. \tag{4.1}$$

This can also be viewed in terms of characters. Fix an orthonormal basis $\{A_1, A_2, A_3\}$ of $\mathfrak{su}(2)$ with $A_1 = \mathrm{diag}(i, -i)$ and consider the maximal torus $T = \{\exp(\theta A_1) | 0 \leqslant \theta \leqslant 2\pi\}$. Each $SU(2)$ representation decomposes into complex one-dimensional irreducible representations L_k of T labeled by their characters $e^{ik\theta}$. The character of the representation $D_{l/2}$ is

$$\chi(D_{l/2}) = e^{il\theta} + e^{i(l-2)\theta} + \cdots + e^{-il\theta} = \frac{e^{i(l+1)\theta} - e^{-il\theta}}{e^{i\theta} - e^{-i\theta}}\,. \tag{4.2}$$

Now fix an odd integer l and consider $D^r_2 \times D_{l/2} = \mathbb{R}^5 \times \mathbb{H}^{l+1/2}$. The restriction to the unit sphere is a trivial vector bundle

$$E_l = S^4 \times \mathbb{H}^{l+1/2}$$

over S^4 with $SU(2)$ action. We can identify D^r_2 with the space $\mathrm{Sym}^2(\mathfrak{su}(2))$ of traceless symmetric endomorphisms of the Lie algebra and hence can consider $x \in S^4$ as a real symmetric matrix x^{ij} on $\mathfrak{su}(2)$ with $g \in SU(2)$ acting by $x \mapsto (\mathrm{Ad}\, g)x(\mathrm{Ad}\, g)^{-1}$. Then

$$H(x) = \sum_{i,j=1}^{3} x^{ij} \rho_l(A_i)\rho_l(A_j) \tag{4.3}$$

is an equivariant section of $\mathrm{End}_{\mathbb{H}}(E_l)$ over S^4 which, moreover, has no degenerate eigenvalues. (See [ASSS].) Hence, E_l decomposes into a direct sum of quaternionic line bundles (the eigenspaces of H). These are the quadrapole bundles.

Each $\rho_l(A_i)$ is (complex) skew-hermitian; so $H(x)$ has real eigenvalues. Thus, each quadrapole bundle is specified by l and the order of its eigenvalue (e.g., "jth from the bottom"). A more convenient labeling scheme is obtained by considering the action of T. The weights of the representation D_2 are $\{0, \pm 2, \pm 4\}$. Thus, T fixes a line; so there are two antipodal fixed points $\{p, -p\}$ on S^4. At these points T acts as an endomorphism of the fiber that commutes with H, splitting each eigenspace into a pair of conjugate complex lines. Such pairs are labeled by the positive weights $\{1, 3, 5, \dots l\}$ of $D_{l/2}$. Thus, to each quadrapole bundle there is a pair of positive odd integers m, n which describe the action of T on the fibers at p and $-p$. We denote this quadrapole bundle by E_{mn}.

Remark. The numbers m, n, and l are related as follows. In the adjoint representation, $\exp(\theta A_1) \in T$ fixes A_1 and rotates the plane $\langle A_2, A_3 \rangle$ by an angle of 2θ. The fixed points are therefore

$$\pm p = \pm \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & & \\ & -1 & \\ & & -1 \end{bmatrix};$$

so $\sqrt{6}H(\pm p) = \pm[3\rho(A_1)\rho(A_1) - \sum \rho(A_i)\rho(A_i)]$. But $\rho_l(A_1) = \mathrm{diag}(1, 3, 5 \dots l)$, and the term involving the summation is $l(l+1)\cdot \mathrm{Id}$; so $H(p)$ is diagonal. Hence, the jth eigenvalue of $H(p)$ (resp. $H(-p)$) corresponds to the eigenvalue $m = 2j - 1$ (resp. $n = l + 2 - 2j$) of $\rho_l(A_1)$. Thus,

$$m + n = l + 1\,. \tag{4.4}$$

Each connection A on $E = E_{mn}$ has two associated operators that are of fundamental importance in gauge theory: the Dirac operator

$$\not{D}_A\colon \Gamma(\Delta^+ \otimes E) \to \Gamma(\Delta^- \otimes E) \tag{4.5}$$

and the self-duality operator

$$D = d_A^* + d_a\colon \Omega^1(Ad\, E) \to \Omega^0(Ad\, E) \oplus \Omega^2_-(Ad\, E)\,. \tag{4.6}$$

(Here, $\Delta_\pm$ are the spin bundles and $\Omega^p(Ad\, E)$ is the space of p-forms with values in the adjoint bundle $Ad\, E = \{L \in \mathrm{Hom}_{\mathbb{H}}(E, E) | L^t = -L\}$.) When A is an equivariant connection, these operators are $SU(2)$-invariant (the $SU(2)$ action on S^4 lifts uniquely to the spin bundle), and hence their indices

$$\mathrm{index}\, D = \ker D - \mathrm{coker}\, D$$

lie in the representation ring of $SU(2)$. In fact, both operators are real; so the indices are formal linear combinations of real representations. We will determine these indices explicitly by applying the G-index theorem of Atiyah and Singer.

As above, the action of T gives the decomposition $D_2^r = 1 \oplus L_2 \oplus L_4$. Fix an orientation on D_2^r. Then the induced orientations on T_pS^4 and $T_{-p}S^4$ are opposite, and hence the normal bundles at the fixed points are

$$N_p = L_2 \oplus L_4 = -N_{-p}. \tag{4.7}$$

Applying the G-spin theorem [AS], we have

$$\chi(\text{index } \not{D})(g) = \frac{\mathrm{ch}_g(E)[p]}{4\sinh(i\theta)\sinh(2i\theta)} - \frac{\mathrm{ch}_g(E)[-p]}{4\sinh(i\theta)\sinh(2i\theta)} \tag{4.8}$$

$$= \frac{(e^{im\theta} + e^{-im\theta}) - (e^{in\theta} + e^{-in\theta})}{(e^{i\theta} - e^{-i\theta})(e^{2i\theta} - e^{-2i\theta})}. \tag{4.9}$$

One of the numbers $(m+n)/2$, $(m-n)/2$ is odd—call it α—and the other is even—call it 2β.

$$= \frac{(e^{i\alpha\theta} - e^{-i\alpha\theta})(e^{2i\beta\theta} - e^{-2i\beta\theta})}{(e^{i\theta} - e^{-i\theta})(e^{2i\theta} - e^{-2i\theta})} \tag{4.10}$$

$$= \frac{(e^{i\alpha\theta} - e^{-i\alpha\theta})}{(e^{i\theta} - e^{-i\theta})}\left[\sum_{s=1}^{\beta} (-1)^{\beta+s+1}\left(\frac{(e^{(2s-1)i\theta} - e^{-(2s-1)i\theta})}{e^{-i\theta} - e^{-i\theta}}\right)\right]. \tag{4.11}$$

This is recognizable as the character of a virtual representation that we will denote $R(\alpha, \beta)$. It is defined for integers α, β, with α odd. From (4.10) we have $R(\alpha, \beta) = -R(-\alpha, \beta) = -R(\alpha, -\beta) = R(2\beta, \alpha/2)$ and $R(\alpha, 0) = 0$, and for $\alpha, \beta > 0$ (4.11) and (4.2) show that

$$R(\alpha, \beta) = D_{(\alpha-1)/2} \otimes [D_{\beta-1} - D_{\beta-2} + D_{\beta-3} - \cdots \pm 1]. \tag{4.12}$$

This can, of course, be written as a formal sum of irreducibles using (4.1).

The self-duality operator is equivalent ([AHS, §6]) to the Dirac operator

$$\not{D}\colon \Gamma(\Delta^+ \otimes \Delta^- \otimes Ad\, E) \to \Gamma(\Delta^- \otimes \Delta^- \otimes Ad\, E). \tag{4.13}$$

Hence its index is given by (4.8) with E replaced by $\Delta^- \otimes Ad\, E$. From (4.7) we have $\Lambda_-^2 N_p = 1 \oplus L_2$ and $\Lambda_-^2 N_{-p} = 1 \oplus L_6$. But $\Delta^- \otimes \Delta^- = 1 \oplus \Lambda_-^2$; so

$$\Delta_p^- = L_1 \oplus L_{-1} \qquad \Delta_{-p}^- = L_3 \oplus L_{-3}.$$

Also noting that $Ad\, E_{\pm p} = \mathrm{Sym}^2\, E_{\pm p}$ as T-representations, we have

$$\chi(\text{index } D)(g) = \frac{\mathrm{ch}_g(\Delta^- \otimes Ad\, E)[p]}{4\sinh(i\theta)\sinh(2i\theta)} - \frac{\mathrm{ch}_g(\Delta^- \otimes Ad\, E)[-p]}{4\sinh(i\theta)\sinh(2i\theta)}$$

$$= \frac{(e^{i\theta} + e^{-i\theta})(e^{2im\theta} + 1 + e^{-2im\theta}) - (e^{3i\theta} + e^{-3i\theta})(e^{2in\theta} + 1 + e^{-2in\theta})}{(e^{i\theta} - e^{-i\theta})(e^{2i\theta} - e^{-2i\theta})}$$

$$= \frac{(e^{ia\theta} - e^{-ia\theta})(e^{2ib\theta} - e^{-2ib\theta})}{(e^{i\theta} - e^{-i\theta})(e^{2i\theta} - e^{-2i\theta})} + \frac{(e^{ic\theta} - e^{-ic\theta})(e^{2id\theta} - e^{-2id\theta})}{(e^{i\theta} - e^{-i\theta})(e^{2i\theta} - e^{-2i\theta})} - 1$$

where $a = m - n - 1$, $2b = m + n + 2$, $c = m - n + 1$, and $2d = m + n - 2$. Using (4.10)–(4.12), we can again express this in terms of the characters of representations $R(\alpha, \beta)$. This establishes the following theorem.

THEOREM 4.1. *For an equivariant connection on the bundle* E_{mn}

$$\text{index } \not{D}_A = R(\alpha, \beta), \tag{4.13}$$

$$\text{index } D_A = R(a, b) + R(c, d) - 1 \tag{4.14}$$

where α, β, a, b, c, *and* d *are as defined above.*

The dimension of these virtual representations can be expressed in terms of the instanton number $k = -c_2(E)$ using the ordinary index theorem. The well-known result is that dim index $\not{D} = k$ and dim index $D = 8k - 3$ ([AHS, §7,10]). Since dim $R(\alpha, \beta) = |\alpha\beta|$, (4.13) and (4.14) both yield

$$k = -c_2(E_{mn}) = \alpha\beta = \frac{m^2 - n^2}{8}. \tag{4.15}$$

Examples

(1) $E_{3,1}$ has $k = 1$, index $\not{D} = R(1, 1) = 1$, and index $D = R(1, 3) + R(3, 1) - 1 = D_2$. (This bundle supports the standard round $k = 1$ self-dual instanton on S^4.)

(2) $E_{5,1}$ has $k = 3$, index $\not{D} = R(3, 1) = D_1$, and index $D = R(3, 4) + R(5, 2) - 1 = D_4 + D_3 + D_2$.

(3) $E_{11,7}$ has $k = 9$, index $\not{D} = D_4$, and index $D = D_{10} + D_9 + D_8 + D_6 - 1$.

COROLLARY 4.2 (see [SS]). *There are no equivariant* SD *or* ASD *connections on* E_{mn} *for* $m, n > 1$, *and the trivial connection is equivariant only on* $E_{1,1}$.

Proof. By reversing orientation if necessary, we may assume that $k \geqslant 0$. Then $m \geqslant n$ and E_{mn} admits no ASD connections. If there is an equivariant SD connection, the vanishing theorem for the self-duality operator on S^4 ([AHS, §6]) shows that coker D_A is the space H_A^0 of covariant constant sections of $Ad\, E$.

For $k = 0$ we have dim $H_A^0 \leqslant 3$ with equality if and only if A is the trivial connection. But (4.15) shows the $k = 0$ case has $m = n$ and index $D = R(-1, m+1) + R(1, m-1) - 1 = -D_m + D_{m-1} - 1$. Since $\dim D_m = 2m + 1$, the only possible SD connection is the trivial connection on $E_{1,1}$. Conversely, the trivial connection occurs only when $k = 0$, and it is then self-dual.

For $k > 0$ all connections are irreducible; so $H_A^0 = 0$ and index D is an actual representation. Now by (4.1) the trivial representation 1 occurs in $D_r \otimes D_t$ if and only if $r = t$. Hence the coefficient of 1 in $R(2r+1, t)$ is the coefficient of D_r in $\sum_{s=0}^{t-1}(-1)^{t+s-1}D_s$, which $(-1)^{t+r+1}$ if $t > r$ and 0 otherwise. It follows that when $n > 1$ the trivial representation occurs with coefficient -1 in (4.14); so the index is not an actual representation. This leaves only the cases $(m, 1)$. □

Bor, Sadun, and Segert have shown that self-dual equivariant connections exist on the remaining bundles $E_{m,1}$ and $E_{1,n}$ ([BSS]).

5. Applications of the Morse theory. The equivariant Morse theory constructed in Section 3 applies to the quadrapole bundles and gives existence results for YM and YMH fields. In fact, we will use only the most immediate consequence of the Morse theory: the existence of a minimum in each component of $\mathscr{B}^H$.

Each quadrapole bundle E_{mn} is a lift of the action of $SU(2)$ on the unit sphere in D_2. This action on S^4 factors through $SO(3)$ and is easily analysed. (See [B, §I.4].) The principal orbits are 3-dimensional, and there are two exceptional orbits each diffeomorphic to $\mathbb{RP}^2$. At a point on a principal orbit, the isotropy group is conjugate to the quaternionic subgroup $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (considering $SU(2)$ as the unit quaternions), and the isotropy representation is the sum $\mathbb{R} \oplus V$, where i, j, k act trivially on $\mathbb{R}$ and as reflections through perpendicular lines in $V \cong \mathbb{R}^3$. Consequently,

(i) all orbits have dimension at least two, and

(ii) a reducible connection on E_{mn} gives a decomposition $E_{mn} = L \oplus \bar{L}$, where L is a complex line bundle. The curvature F of L is then a real-valued 2-form invariant under the isotropy subgroup. But at a point x in a principal orbit

$$(\Lambda^2(T_x^*S^4))^Q \cong (V \oplus \Lambda^2 V)^Q \cong V^Q \oplus V^Q = \{0\};$$

so F vanishes at such x and hence everywhere. Thus, the only reducible connection is the trivial connection, and this appears only on E_{11} by Corollary 4.2.

Now fix a principal bundle P over S^4 with instanton number $k = -c_2(P)$. For each positive odd factor α of $k \neq 0$, we get, as in (4.10) and (4.15), an associated pair of integers m, n and a quadrapole bundle E_{mn}. This specifies a lift of the above $SU(2)$ action to P, different α give different lifts, and there are no reducible equivariant connections. For $k = 0$ there is a distinct quadrapole bundle E_{mm} for each positive odd integer, and the only reducible equivariant connection is the trivial connection of $E_{1,1}$. Thus, excluding the (1, 1) bundle, (3.3) reads

$$\mathscr{B}^H = \bigcup_{\alpha/k} \mathscr{B}_\alpha \quad \text{for } k \neq 0, \qquad \mathscr{B}^H \supset \bigcup_{m>1} \mathscr{B}_{mm} \quad \text{for } k = 0. \tag{5.1}$$

Corollary 4.2 shows that, for $k \neq \pm 1$, $\mathscr{M}^H \cap \mathscr{B}^H_\alpha = \emptyset$ for at least one α. Thus, Theorem 3.4 (or direct minimization on each $\mathscr{B}_\alpha$) immediately gives a result of Sadun and Segert.

COROLLARY 5.2 ([SS]). *Every principal $SU(2)$ bundle P over S^4 with $c_2(P) \neq \pm 1$ admits an irreducible nonminimal Yang-Mills field.* ■

The $SU(2)$ actions specified by the quadrapole bundles induce actions on each vector bundle $E \to M$ associated to P, giving a decomposition like (3.4) covering (5.1). Again, the Morse theory immediately gives critical points, but now we must take care that these are not the uncoupled solutions described in Section 2. We will call E *nontrivial* if it is associated to P by a representation that contains no invariant lines.

COROLLARY 5.3. *Let P be a principal $SU(2)$ bundle over S^4 and let E be a nontrivial associated bundle. Then there is a Yang-Mills-Higgs field on (P, E). This YMH field is not one of the uncoupled solutions of Section 2, and its connection is irreducible.*

Proof. By Theorem 3.1 there exists a minimum of the YMH function on each component of $\mathscr{E}^H$, and this is a YMH field on a quadrapole bundle. We will show that it is nontrivial (i.e. $\phi \neq 0$) when the mass parameter μ is large.

Choose $A \in \mathscr{B}^H$ and set $C = YM(A)$. Then

$$\mathscr{M}^H_C = \{[A] \in \mathscr{B}^H \mid A \text{ is Yang-Mills and } YM(A) \leqslant C\}$$

is nonempty and compact because YM satisfies Condition C on $\mathscr{B}^H$. Let Δ_A denote the Laplacian $\nabla^*\nabla$ of the connection A acting on $L^{1,2}_H(E)$. The eigenvalues of Δ_A are continuous functions of the connection; so there is a constant M such that the first eigenvalue satisfies $\lambda_1(\Delta_A) \leqslant M \ \forall A \in \mathscr{M}^H_C$. Choose $\mu > M/2$. Then at each $A \in \mathscr{M}^H$ the first eigenfunction ψ of Δ_A satisfies

$$\text{Hess } YMH_{(A,0)}(\psi, \psi) = \int_{S^4} |\nabla^A \psi|^2 - 2\mu|\psi|^2 = (\lambda_1 - 2\mu)\int_{S^4} |\psi|^2 < 0.$$

Thus, the fields $(A, \phi) \in \mathscr{E}$ with $\phi \equiv 0$ are never minimal in $\mathscr{E}^H$. ■

By Proposition 2.1 these fields are all unstable critical points of the YMH function; so the action, although minimal amongst equivariant fields, can be reduced by perturbations in nonequivariant directions.

APPENDIX

This appendix presents the proof of Theorem 2.3. In the Yang-Mills case ($\phi \equiv 0$) this is due to K. Uhlenbeck, with refinements by S. Sedlacek and C. Taubes. The extension to the Higgs case requires only a technical addition to the existing argument (Lemma A.2 below) and the removable singularity theorem of [Pk1].

Let $\{A^i, \phi^i\}$ be a Palais-Smale sequence for YMH on $\mathscr{E}$ with its $L^{1,2}$ Riemannian metric (2.2). This means that $YMH(A^i, \phi^i) < C$ and

$$\|\text{grad } YMH\|_{1,2,A^i} = \|d(YMH)\|_{-1,2,A^i} \to 0 \tag{A.1}$$

where the differential $d(YMH)$ is, from (2.1),

$$d(YMH)_{(A,\phi)} = (d^*F + \text{Re}\langle\nabla\phi, \rho(\ )\phi\rangle, \nabla^*\nabla\phi + (|\phi|^2 - \mu)\phi). \tag{A.2}$$

While our aim is a global convergence theorem, the proof proceeds by choosing a cover $\{U_\alpha\}$ of M, showing convergence on each U_α, and showing that the limit fits together to define a global limit. The first step is a covering lemma. The proof is exactly as given by Sedlacek [S].

LEMMA A.1. *For each $\kappa > 0$ there exists a finite set of points $\{x_l\} \subset M$, a subsequence $\{A^i\}$, and a countable cover $\{U_\alpha\}$ of $M - \{x_l\}$ by geodesic balls such that*

$$\liminf \int_{U_\alpha} |F^i|^2 + |d^i\phi^i|^2 + |\phi^i|^4 < \kappa^2 \qquad \forall\alpha. \tag{A.3}$$

When κ is small, we can apply [U, Theorem 1.3] to obtain, for each i and α, a $L^{2,2} \cap L^{1,5}$ trivialization τ_α^i of P over U_α such that the connection form a^i of A^i in this trivialization satisfies

$$d^*a^i = 0 \qquad \text{and} \qquad \|a^i\|_{1,2} \leqslant c_1\|F^i\|_2 \leqslant c_1\kappa \qquad \text{on } U_\alpha. \tag{A.4}$$

(Here and below, c_i denotes a constant depending only on the Riemannian metric on M.) The $\{\tau_\alpha^i\}$ also trivialize E. Thus, on each U_α we have a sequence $\{a^i\}$ of $\mathfrak{g}$-valued 1-forms and a sequence $\{\phi^i\}$ of $\mathbb{R}^n$-valued functions.

LEMMA A.2. *For each α there is a subsequence $\{a^i, \phi^i\}$ that converges in $L^{1,2}_{\text{loc}}(U_\alpha)$.*

Proof. Fix α and $x \in U = U_\alpha$. Since the action is bounded and gauge invariant, the sequence $\{(a^i, \phi^i)\}$ is bounded in $L^{1,2}(U)$. Hence, there is a subsequence that converges weakly in $L^{1,2}$ and strongly in L^2. By (A.4) the limit $(a, \phi) \in L^{1,2}$ satisfies $d^*a = 0$ and $\|(a, \phi)\|_{1,2} \leqslant \underline{\lim}\,\|(a^i, \phi^i)\|_{1,2} \leqslant c_2\kappa$. Set $\xi^i = (a^i - a, \phi^i - \phi)$. Then $\{\xi^i\}$ is bounded in $L^{1,2}$, $\xi^i \to 0$ in L^2, and we must show that $\xi^i \to 0$ in $L^{1,2}_{\text{loc}}$.

Fix a ball $B = B(x, \delta)$ with $B_2 = B(x, 2\delta) \subset U$ and fix a smooth bump function $0 \leqslant \beta \leqslant 1$ supported on B_2 with $\beta \equiv 1$ on B and $|d\beta| \leqslant 4/\delta$. Then

$$\begin{aligned}
\|\xi^i\|^2_{1,2;B} &\leqslant c_2 \int_B |d\xi^i|^2 + |d^*a^i|^2 + |\xi^i|^2 \\
&\leqslant c_2 \int_U \beta|d\xi^i|^2 + |\xi^i|^2 \qquad \text{(A.5)}\\
&\leqslant c_2 \int_U \langle\xi^i\, d^*\, d\xi^i\rangle - \langle d\beta \wedge \xi^i, d\xi^i\rangle + |\xi^i|^2.
\end{aligned}$$

For large i the last two terms are bounded by

$$\left(\|\xi^i\|_2 + \frac{4}{\delta}\|\xi^i\|_{1,2}\right)\|\xi^i\|_2 \leqslant \left(1 + \frac{c_2\kappa}{\delta}\right)\|\xi^i\|_2. \tag{A.6}$$

In the gauge τ_α^i we have $d^i = d + a^i$, and (with abbreviated notation)

$$\langle \xi^i, d^* d\xi^i\rangle = \langle \xi^i, (d^i)^* d^i\xi^i + 2a^i\cdot d\xi^i - (d^* a^i)\cdot \xi^i + a^i\cdot a^i\cdot \xi^i\rangle. \tag{A.7}$$

Using the Holder and Sobolev inequalities, each of the last three terms in this expression are bounded by $c_3\kappa\ \|\xi^i\|_{1,2}^2$ for some c_3.

Given $\varepsilon > 0$, we can combine (A.5)–(A.7), use the weak convergence $\xi^i \rightharpoonup 0$, and choose L large enough that

$$\|\xi^i\|_{1,2,B}^2 \leqslant \varepsilon + c_3\kappa\|\xi^i\|_{1,2}^2 + \int \langle \xi^i, (d^i)^* d^i(a^i, \phi^i)\rangle \qquad \forall i > L. \tag{A.8}$$

Now $F^i = d^i a^i - a^i \cdot a^i$; so using (A.2), we have

$$(d^i)^* d^i(a^i, \phi^i) = (d^i)^* F^i + (d^i)^* d^i\phi^i - (d^i)^* d^i(a^i\cdot a^i) = d(YMH)^i - B^i$$

where

$$B^i = d^i\phi^i\cdot\phi^i + (|\phi^i|^2 - \mu)\phi^i + (d^i)^* d^i(a^i\cdot a^i). \tag{A.9}$$

Since $L^{1,2}$ and $L^{-1,2}$ are dual spaces, (A.1) gives

$$\int \langle \xi^i, d(YMH)^i\rangle \leqslant \|\xi^i\|_{1,2,a^i}\cdot\|d(YMH)^i\|_{-1,2,a^i} < \varepsilon \qquad \forall i > L'.$$

By repeatedly substituting $d^i = d + a^i$ and $(a^i, \phi^i) = \xi^i + (a, \phi)$, (A.9) can be written as a sum of terms of three types: (i) terms of the form $\xi^i A$, where $A \in L^{1,2}$ is independent of i, which therefore have vanishing limit since $\xi^i \rightharpoonup 0$, (ii) terms of the form $\xi^i B \xi^i$, which can be bounded by $c_4\kappa\|\xi^i\|_{1,2}^2$, and (iii) the term $\mu|\xi^i|^2$, which has limit 0. Thus,

$$\|\xi^i\|_{1,2} \leqslant 3\varepsilon + c_5\kappa\|\xi^i\|_{1,2} \qquad \forall i > L''. \tag{A.10}$$

Because this constant c_5 is universal, we can use Lemma A.1 to choose κ at the beginning so that $c_5\kappa < 1/2$. Then (A.10) shows that $\xi \to 0$ in $L^{1,2}$. ■

Applying Lemma A.2 to each U_α and taking a diagonal subsequence yields a subsequence that converges in $L^{1,2}$ on each U_α to a limit (a_α, ϕ_α) with $d^* a_\alpha = 0$. On each nonempty $U_\alpha \cap U_\beta$ the transition functions defined by $\tau^i_\beta = g^i_{\alpha\beta}\tau^i_\alpha$ satisfy

$$a_\alpha = g_{\alpha\beta} a_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta}\, dg_{\alpha\beta}^{-1} \tag{A.11}$$

for each i. A bootstrap argument (see [S, Lemma 3.5]) and a clever estimate of Taubes ([T1, Lemma A1]) then give the convergence of the transition functions in $L^{2,2} \cap C^0$. The limit $\{g_{\alpha\beta}\}$ preserves the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

On each U_α the limit connection is a weak solution to $d^*a = 0$ and the YMH equations. Together, these are an elliptic system, and standard bootstrap arguments then show that $a \in C^\infty$ (see [Pk1]). Then (A.11) implies that the $\{g_{\alpha\beta}\}$ are smooth. Hence, the data $\{U_\alpha, \gamma_{\alpha\beta}, a_\alpha\}$ define a smooth YMH field on a smooth bundle over $M - \{x_l\}$. The C^0 convergence $g^i_{\alpha\beta} \to g_{\alpha\beta}$ implies that this bundle is isomorphic to P over $M - \{x_l\}$ (Lemma 3.2 of [U]). Thus, we have found a subsequence of the original sequence on P that convergence in $L^{1,2}(K)$ for $K \subset\subset M - \{x_l\}$ to a smooth YMH field on $M - \{x_l\}$. Finally, by the removable singularity theorem ([Pk1]), this extends to a smooth YMH field on a new bundle P'.

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