Geometric Analysis   Problem Set 2

Due Monday, February 8

Problem (2.1) Re-prove the second Bianchi identify as stated in Proposition 0.6 this time using a useful frame (take X, Y, Z to be basis vectors in the frame and choose ξ with $(\nabla\xi)_p = 0$).

Problem (2.2) Prove the formula $\nabla^* = -\sum_{ij} g^{ij}\, e_i \lrcorner \nabla_{e_j}$ by first showing that for any $\xi \in \Gamma(E)$ and $\eta \in \Gamma(T^*M \otimes E)$ the $(n-1)$-form $\omega = \sum \langle \eta, e^i \otimes \xi \rangle\, e_i \lrcorner dv_g$ is well-defined (i.e. is independent of the frame), then computing $d\omega$ in a useful frame, and integrating.

Problem (2.3) Let ∇ be a connection on a (real) bundle E that is compatible with the metric $\langle\ ,\ \rangle$ on E. Show that $\langle \phi, \nabla^*\nabla\phi \rangle = |\nabla\phi|^2 + \frac{1}{2} d^* d|\phi|^2$ for all $\phi \in \Gamma(E)$, and consequently

$$\int_M \langle \phi, \nabla^*\nabla\psi \rangle \ = \ \int_M \langle \nabla\phi, \nabla\psi \rangle$$

for all smooth compactly suported sections $\phi, \psi \in \Gamma(E)$.

Problem (2.4) Complete the proof of Proposition 3.6 (in the lecture notes) by proving that any second order LDO $D : \Gamma(E) \to \Gamma(F)$any second order LDO can be written

$$D \ = \ \sigma_D \circ \nabla^2 + B \circ \nabla + C$$

where $B : T^*M \otimes E \to F$ and $C : E \to F$ are bundle maps. Then state and sketch the proof of the corresponding formula for k^{th} order operators.

Problem (2.5) Let $D : \Gamma(E) \to \Gamma(F)$ and $\tilde{D} : \Gamma(F) \to \Gamma(G)$ be linear differential operators with symbols σ_D and $\sigma_{\tilde{D}}$ and orders k and $\tilde{k}$.

(a) Let M_f denote multiplication by $f \in C^\infty(M)$. Show that $[D, M_f]$ is an LDO of order $k-1$.
(b) Show that $\sigma_{D\circ\tilde{D}} = \sigma_D \circ \sigma_{\tilde{D}}$.
(c) Show that $\sigma_{D^*} = (-1)^k (\sigma_D)^*$.