# Geometric Analysis   Problem Set 3

Due Monday, February 22

**Problem (3.1)** This problem refers to the online course notes. Start by reading from the bottom of page 15 to the end of Section 2.

Using the notation $a_i$ and $a^{*i}$ introduced before (2.21) show that, for each $i$, $a^{*i}a_i$ is an even derivation of $\Lambda^*(V^*)$ that vanishes on $\Lambda^0(V^*)$. Then use Lemma 2.6b and equations (2.21) to prove the formulas in parts (a), (b) and (c) of Example 2.8.

**Problem (3.2)** Prove that, for the Sobolev spaces of functions on a 4-manifold, multiplication induces a bounded linear map $L^{3,2} \times L^{2,2} \to L^{2,2}$ (this is one instance of the Multiplication Theorem). Use only the definition of the Sobolev norm and the Sobolev Embedding Theorem.

**Problem (3.3)** Prove the Interpolation Inequality by the method used in class to prove the Poincaré Inequality.

Due Wednesday, February 24

**Problem (3.4)** Suppose that $f$ is a $W^{1,2}$ weak solution of

$$\Delta f \;=\; \left(\sqrt{1+|df|^2}\right) f$$

on a compact Riemannian 3-manifold. Prove that $f$ is smooth using the bootstrap procedure listed below. For the various steps, apply the Sobolev Embedding Theorem, the Elliptic Regularity Theorem (first in $W^{k,p}$ form, then $C^{k,\alpha}$ form), Holder's inequality, and perhaps the multiplication and interpolation theorems.

(a) Show that $f$ lies in $L^6$, that $g = \sqrt{1+|df|^2}$ lies in $L^2$, and that $gf \in L^{3/2}$.

(b) Show that $f \in L^{2,3/2}$, $g \in L^3$, and $gf \in L^{5/2}$.

(c) Show that $gf \in C^\alpha$ for some $\alpha > 0$.

(d) Show that $f \in C^\infty$.