Math 851: Numerical Analysis II
Homework Assignment One

Spring 2009

1. Define f: $R^2 \to R^1$ by $f(x) = x_1$ if $x_2 = 0$, $f(x) = x_2$ if $x_1 = 0$, and $f(x) = 1$ otherwise. Show that the partial derivatives $\frac{\partial f}{\partial x_1}(0)$ and $\frac{\partial f}{\partial x_2}(0)$ exist, but that f does not have a G-derivative at 0.

2. Let $A \in L(R^n, R^n)$ and define $f : R^n \to R^1$ by $f(x) = x^T Ax$. Show that f has an F-derivative at each $x \in R^n$ and compute $f'(x)$.

3. Define $f : R^2 \to R^1$

$$f(x) = \begin{cases} 0 & \text{if } x_1 = 0 \\ \frac{2x_2 \exp(-x_1^{-2})}{(x_2^2 + \exp(-2x_1^{-2}))} & \text{if } x_1 \neq 0 \end{cases} \tag{0.1}$$

   Show that f has a G-derivative at 0, but that f is not continuous at zero.

4. Let $F : R^2 \to R^2$ be defined by $f_1(x) = x_1^3$, $f_2(x) = x_2^2$. Set $x = (0, 0)$ and $y = (1, 1)^T$. Show that there is no z in the segment defined by x and y such that

$$F(y) - F(x) = F'(z)(y - x). \tag{0.2}$$

5. Let $A \in L(R^n, R^n)$ be symmetric, and define $g : R^n \to R^1$ by $g(x) = x^T Ax$. Show that g is convex if and only if A is positive semi-definite and g is both strictly and uniformly convex if and only if A is positive definite.

6. Show that $g : D \subset R^n \to R^1$ is convex on the convex set if and only if for any $x \in D$, $y \in D$, the function $f : [0, 1] \to R^1$, $f(t) = g(tx + (1 - t)y)$, is convex on $[0, 1]$.

7. Assume that $F : D \subset R^n \to R^n$ has a G-derivative which satisfies $\|F'(x)\| \leq \alpha < 1$ for all x in a convex set $D_0 \subset D$. Show that F is contractive on D_0.

8. Define $f : [0, 1] \subset R^1 \to R^1$ by $f(x) = \frac{1}{2}x + 2$, $x \in [0, 1]$. Show that f is contractive on [0,1], but has no fixed point.

Due date: Wednesday, Jan. 28, 2009. In class