1. (5 points) **True or False** (Circle T or F, No explaination is needed) ?

T F $\begin{bmatrix} 1 & 0 & 0 & 20 & 17 \\ 0 & 0 & 1 & 2 & 15 \end{bmatrix}$ is in reduced row echelon form.

T F $(A+B)^2 + (A-B)^2 + 2(A+B)(A-B) = 4A^2$ holds for any $n \times n$ matrices A and B.

T F Any nonzero scalar multiple of an elementary matrix is an elementary matrix.

1pt for everyone ~~T F $\det(\mathbf{x}\mathbf{y}^{\mathrm{T}}) = 0$ holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.~~ $n > 2$

T F Given an $m \times n$ matrix A with n pivots, $\mathbf{x} = \mathbf{0}$ is the only solution of $A\mathbf{x} = \mathbf{0}$.

2. (25 points) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

(a) (15 points) Find the inverse of the matrix A.

Sol: $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \underset{R_3 - 3R_1}{\overset{R_2 - 2R_1}{\approx}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -1 & -3 & 0 & 1 \end{array}\right)$

$\overset{R_3 - 2R_2}{\approx} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \underset{R_1 - R_3}{\overset{R_2 + R_3}{\approx}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$

Thus $A^{-1} = \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$ #

(b) (10 points) Compute $\det(A^{-8})$

Sol: $\det A = 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 0 + 4 - 3 = 1$

$\det(A^{-8}) = \dfrac{1}{(\det A)^8} = \dfrac{1}{1^8} = 1$

3. (25 points) Consider the system of linear equations

$$\begin{array}{rcrcrcl} x & - & y & + & z & = & a \\ x & - & 3y & + & 3z & = & b \\ 2x & + & y & - & dz & = & c \end{array}$$

(a) (15 points) Find the condition on a, b, c, and d such that the system will have infinitely many solutions.

Sol:
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & -3 & 3 & b \\ 2 & 1 & -d & c \end{array}\right)
\overset{R_2-R_1}{\underset{R_3-2R_1}{\sim}}
\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & -2 & 2 & b-a \\ 0 & 3 & -d-2 & c-2a \end{array}\right)$$

$$\overset{R_2/-2}{\underset{R_3/3}{\sim}}
\left(\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & -1 & \frac{b-a}{-2} \\ 0 & 1 & \frac{-d-2}{3} & \frac{c-2a}{3} \end{array}\right)$$

$$\overset{R_1+R_2}{\underset{R_3-R_2}{\sim}}
\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2}a-\frac{b}{2} \\ 0 & 1 & -1 & \frac{1}{2}a-\frac{1}{2}b \\ 0 & 0 & \frac{1-d}{3} & -\frac{7}{6}a+\frac{1}{2}b+\frac{1}{3}c \end{array}\right)$$

The system admits ∞ sol's

$\Longleftrightarrow \frac{1-d}{3} = -\frac{7}{6}a+\frac{1}{2}b+\frac{1}{3}c = 0$

$\Longleftrightarrow \begin{cases} d = 1 \\ 7a-3b-2c = 0 \end{cases}$

#

(b) (10 points) Solve the linear system when $a = 2, b = 0, c = 1, d = 7$.

Sol: Plug $a=2$ $b=0$ $c=1$ $d=7$ into (*), yielding

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right)
\overset{R_3/-2}{\sim}
\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)
\overset{R_2+R_3}{\sim}
\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Thus the system admits only one sol.

$(x, y, z) = (3, 2, 1)$

#

4. (15 points) Let $U = \{f \in C^1[0,1] \mid f(0) = f'(0) = 0\}$. Show that U is a subspace of $C^1[0,1]$.

Proof: $\forall f, g \in U$, $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in C^1[0,1]$)

$$(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

$$(\alpha f + \beta g)'(0) = \alpha f'(0) + \beta g'(0) = \alpha 0 + \beta \cdot 0 = 0$$

$\Rightarrow$ $\alpha f + \beta g \in U$

Therefore U is a subspace of $C^1[0,1]$.

#

5. (15 points) Show that the functions x, $\sin(x)$, xe^x, 1 are linearly independent in $C(-\infty, \infty)$.

Proof: The Wronskian

$$W[1, x, \sin(x), xe^x](x) = \begin{vmatrix} 1 & x & \sin(x) & xe^x \\ 0 & 1 & \cos(x) & (x+1)e^x \\ 0 & 0 & -\sin(x) & (x+2)e^x \\ 0 & 0 & -\cos(x) & (x+3)e^x \end{vmatrix}$$

$$= (x+2)e^x \cos(x) - (x+3)e^x \sin(x)$$

Let $x = 0$, we have

$$W[1, x, \sin(x), xe^x](0) = 2e^0 \cdot 1 - 0 = 2 \neq 0$$

Thus, x, $\sin(x)$, xe^x, 1 are LI in $C(-\infty, \infty)$

#

6. (15 points) Suppose that V is a vector space, k is a positive integer and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is a linearly independent set of vectors in V. We define, for $i = 1, \cdots, k$,

$$\mathbf{u}_i := \mathbf{v}_i + \mathbf{v}_{i+1}.$$

Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_{k+1}\}$ is also a linearly independent set of vectors.

Proof: $\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_k \vec{u}_k + \alpha_{k+1} \vec{V}_{k+1} = \vec{0}$

$\Leftrightarrow \sum_{i=1}^{k} \alpha_i (\vec{V}_i + \vec{V}_{i+1}) + \alpha_{k+1} \vec{V}_{k+1} = \vec{0}$

$\Leftrightarrow \sum_{i=1}^{k} \alpha_i \vec{V}_i + \sum_{i=1}^{k} \alpha_i \vec{V}_{i+1} + \alpha_{k+1} \vec{V}_{k+1} = \vec{0}$

$\Leftrightarrow \sum_{i=1}^{k+1} \alpha_i \vec{V}_i + \sum_{i=2}^{k+1} \alpha_{i-1} \vec{V}_i = \vec{0}$

$\Leftrightarrow \alpha_1 \vec{V}_1 + \sum_{i=2}^{k+1} (\alpha_i + \alpha_{i-1}) \vec{V}_i = \vec{0}$

$\vec{V}_i$ LI

$\Leftrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_i + \alpha_{i-1} = 0 \quad i = 2, \ldots, k+1 \end{cases}$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_k \\ \alpha_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$\det A = 1$

$\Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{k+1} = 0$

Thus, $\{\vec{u}_1, \vec{u}_2, \ldots \vec{V}_{k+1}\}$ are LI in V.

7. (10 points) **(Bonus)** In the setting of Problem 6, we further define $\mathbf{u}_{k+1} := \mathbf{v}_{k+1} + \mathbf{v}_1$. Is $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ linearly independent? Please justify your answer.

Sol:
$$\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_{k+1} \vec{u}_{k+1} = \vec{0}$$
$$\Updownarrow$$
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}}_{B} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_k \\ \alpha_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$$\det B = 1 + (-1)^{(k+1)+1} 1 = 1 + (-1)^k = \begin{cases} 0 & k \text{ odd} \\ 2 & k \text{ even} \end{cases}$$

Thus, $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_{k+1}\}$ are $\begin{cases} LI & \text{if } k \text{ even} \\ LD & \text{if } k \text{ odd} \end{cases}$