1. (5 points) **True or False** (Circle T or F, No explaination is needed) ?

T F If $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis of a vector space, so is $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \cdots, \mathbf{v}_{n-1} - \mathbf{v}_n, \mathbf{v}_n\}$.

T F If A is an $n \times m$ matrix, then $nullity(A) + rank(A) = n$.

T F If two $n \times n$ matrices A and B are similar, then $A + I_n$ and $B + I_n$ are also similar.

T F If A is a non-singular $n \times n$ matrix, then A is similar to $\det(A) \cdot I_n$.

T F If $L : V \to W$ is an isomorphism, then $L^{-1} : W \to V$ exists and is also an isomorphism.

2. (20 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the the vectors

$$\mathbf{u}_1 = (2,1,1,0)^T, \quad \mathbf{u}_2 = (1,4,1,1)^T, \quad \mathbf{u}_3 = (3,4,2,1)^T.$$

(a) (15 points) Find a basis for S and determine its dimension.

Sol: $$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 1 & 1 \\ 3 & 4 & 2 & 1 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\approx} \begin{pmatrix} 1 & 4 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{pmatrix} \underset{R_3 - 3R_1}{\overset{R_2 - 2R_1}{\approx}} \begin{pmatrix} 1 & 4 & 1 & 1 \\ 0 & -7 & -1 & -2 \\ 0 & -8 & -1 & -2 \end{pmatrix}$$

$$\overset{R_2 - R_3}{\approx} \begin{pmatrix} 1 & 4 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -8 & -1 & -2 \end{pmatrix} \underset{R_3 + 8R_2}{\overset{R_1 - 4R_2}{\approx}} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 \end{pmatrix} \underset{-R_3}{\overset{R_1 + R_3}{\approx}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Thus, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis of S

and $\dim S = 3$.

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(b) (5 points) Expand the basis for S from Part (a) to a basis of $\mathbb{R}^4$.

Sol: $$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 \end{pmatrix} = 1 \neq 0$$

Therefore, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

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3. (25 points) Let

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 4 \\ 5 & 5 & 10 \end{bmatrix}$$

(a) (8 points) Find a basis for $N(A)$.

Sol: $A \overset{R_3-R_1-R_2}{\approx} \begin{pmatrix} 2 & 4 & 6 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1/2}{\approx} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_2-3R_1}{\approx} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{pmatrix}$

$\overset{R_2/-5}{\approx} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1-2R_2}{\approx} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{\Delta}{=} B$

$N(A) = N(B) = \{\vec{x} \mid B\vec{x} = \vec{0}\} = \{(\alpha, \alpha, -\alpha)^T \mid \alpha \in \mathbb{R}\} = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$

Thus, $\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$ is a basis of $N(A)$ #

(b) (7 points) Find a basis for the row space of A.

Sol: Row space of A = Row Space of B

$= \text{span}\{(1,0,1), (0,1,1)\}$

$(1,0,1)$ & $(0,1,1)$ are linearly independent

$\Rightarrow \{(1,0,1), (0,1,1)\}$ is a basis of the row space of A. #

(c) (8 points) Find a basis for the column space of A.

Sol: dim{Column Space of A} = dim{Row space of A} = 2

Column Space of $A = \text{span}\{\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 10 \end{pmatrix}\}$

$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ & $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ are linearly independent

$\Rightarrow \{\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}\}$ is a basis of the column space of A. #

(d) (2 points) Find rank(A).

Sol: rank(A) = dim (Row space of A) = 2 #

4. (20 points) Consider the basis $B = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ of $\mathbb{R}^3$ where

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(a) (10 points) Find the transition matrix from the standard basis to B.

Sol: $U_{BE} = U_{EB}^{-1}$

$$= \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}^{-1}$$

$$(U_{EB} \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{R_3 - 2R_2}{\overset{R_1 - 2R_2}{\approx}} \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 \end{array}\right)$$

$$\underset{R_2 \leftrightarrow R_3}{\overset{R_1 \leftrightarrow R_2}{\approx}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 & -2 & 0 \end{array}\right) \overset{R_3 - R_2}{\approx} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\overset{R_1 - R_3}{\approx} \left(\begin{array}{c|ccc} & -1 & 1 & 1 \\ I_3 & 0 & -2 & 1 \\ & 1 & 0 & -1 \end{array}\right)$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

(b) (10 points) If $\mathbf{v}_1 = (2, 2, 2)^T$ and $\mathbf{v}_2 = (3, 2, 1)^T$, compute $[\mathbf{v}_1]_B$ and $[\mathbf{v}_2]_B$.

Sol:
$$[\vec{v}_1]_B = U_{BE}\vec{v}_1 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

$$[\vec{v}_2]_B = U_{BE}\vec{v}_2 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$$

5. (30 points) We introduce the vector space V, admitting

$$C = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}, \qquad D = \{e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}\}$$

respectively as its basis, and **complex** numbers as its scalars.
(NOTE: $e^{ix} = \cos(x) + i\sin(x)$ and $i^2 = -1$)

(a) (15 points) Find the matrix A for the linear operator $L = \partial_x^2 + \partial_x$ in the basis of C for V.

Sol: $L(\sin x) = -\sin x + \cos x \Rightarrow [L(\sin x)]_C = (-1, 1, 0, 0)^T$

$L(\cos x) = -\cos x - \sin x \Rightarrow [L\cos x)]_C = (-1, -1, 0, 0)^T$

$L(\sin(2x)) = -4\sin(2x) + 2\cos(2x) \Rightarrow [L(\sin(2x))]_C = (0, 0, -4, 2)^T$

$L(\cos(2x)) = -4\cos(2x) - 2\sin(2x) \Rightarrow [L(\cos 2x)]_C = (0, 0, -2, -4)^T$

Thus, $A = \left([L(\sin x)]_C \; [L(\cos x)]_C \; [L(\sin(2x))]_C \; [L(\cos(2x))]_C\right) = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 2 & -4 \end{pmatrix}$ #

(b) (10 points) Find the transition matrix U from the basis D to C.

Sol: $U = U_{CD} = \left([e^{ix}]_C \; [e^{-ix}]_C \; [e^{2ix}]_C \; [e^{-2ix}]\right)$

$$= \begin{pmatrix} i & -i & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & i & -i \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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(c) (5 points) Write B, the matrix for L in the basis D, in terms of A and U. (You do not need to simplify.)

Sol: $B = U^{-1} A U$

6. (10 points) **(Bonus)** A projection $P : V \to V$ is a linear transformation from a vector space V into itself, satisfying $P^2 = P$. Show that

$$\mathrm{Ker}(P) \cap \mathrm{Im}(P) = \{\mathbf{0}\}, \qquad \mathrm{Ker}(P) + \mathrm{Im}(P) = V.$$

(Note that $L^2 = L \circ L$ and $\mathrm{Ker}(P) + \mathrm{Im}(P) := \{v + w \mid v \in \mathrm{Ker}(P), w \in \mathrm{Im}(P)\}$)

Proof: Let $\vec{v} \in \ker(P) \cap \mathrm{Im}(P)$

$$\left.\begin{array}{l} \vec{v} \in \mathrm{Im}(P) \Rightarrow \exists\, \vec{u} \in V \ni P\vec{u} = \vec{v} \\ \vec{v} \in \ker(P) \Rightarrow P\vec{v} = \vec{0} \end{array}\right\}$$

$\Rightarrow \vec{v} = P\vec{u} = P^2\vec{u} = P(P\vec{u}) = P\vec{v} = \vec{0}$

Thus, $\ker(P) \cap \mathrm{Im}(P) = \{\vec{0}\}$

$\forall\, \vec{v} \in V$. $P^2\vec{v} = P\vec{v} \Leftrightarrow (P^2 - P)\vec{v} = \vec{0}$

$\Leftrightarrow P(Id - P)\vec{v} = \vec{0}$

$\Leftrightarrow (Id - P)\vec{v} \in \ker(P)$

$\Leftrightarrow \exists\, \vec{w} \in \ker(P) \ni$

$\vec{v} - P\vec{v} = \vec{w}$

That is, $\vec{v} = P\vec{v} + \vec{w}$, where $P\vec{v} \in \mathrm{Im}(P)$, $\vec{w} \in \ker(P)$

In other words, $\vec{v} \in \ker(P) + \mathrm{Im}(P)$

$$\left.\begin{array}{l} \text{Therefore, } V \subseteq \ker(P) + \mathrm{Im}(P) \\ \text{Meanwhile, } \ker(P) + \mathrm{Im}(P) \subseteq V \end{array}\right\}$$

$\Rightarrow \ker(P) + \mathrm{Im}(P) = V$

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