§5.6

2. (a) Let $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Gram-Schmidt process:

① $\vec{e}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

② $$\vec{W}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{e}_1 \rangle \vec{e}_1$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\vec{e}_2 = \frac{\vec{W}_2}{\|\vec{W}_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus $Q = (\vec{e}_1 \ \vec{e}_2) = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$R = Q^T A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 \\ 0 & 8 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$ #

(b) Let $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

Gram-Schmidt process

① $\vec{e}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

② $\vec{W}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{e}_1 \rangle \vec{e}_1 = \begin{pmatrix} 5 \\ 10 \end{pmatrix} - 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

$\vec{e}_2 = \dfrac{\vec{W}_2}{\|\vec{W}_2\|} = \dfrac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Thus $Q = (\vec{e}_1 \ \vec{e}_2) = \dfrac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$R = Q^T A = \dfrac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix} = \dfrac{1}{\sqrt{5}} \begin{pmatrix} 5 & 20 \\ 0 & 15 \end{pmatrix} = \sqrt{5} \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$ #

7 Sol: $\|\vec{x}_1\| = \frac{1}{2}\sqrt{1^2+1^2+1^2+(-1)^2} = 1$ $\quad \|\vec{x}_2\| = \frac{1}{6}\sqrt{1^2+1^2+3^2+5^2} = 1$

$\langle \vec{x}_1, \vec{x}_2 \rangle = \frac{1}{12}(1\cdot1+1\cdot1+1\cdot3-1\cdot5) = 0$

$\Rightarrow \{\vec{x}_1, \vec{x}_2\}$ is <u>ON</u>

Let $A = \begin{pmatrix} \vec{x}_1^T \\ \vec{x}_2^T \end{pmatrix}$ we solve $A\vec{x} = \vec{0}$

$$A \sim \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 1 & 0 & -4 \\ 0 & 0 & \boxed{1} & 3 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 - 4x_4 = 0 \\ x_3 + 3x_4 = 0 \end{cases}$$

Let $x_2 = 1$ $x_4 = 0 \Rightarrow x_1 = -1$ $x_3 = 0$

Let $x_2 = 0$ $x_4 = 1 \Rightarrow x_1 = 4$ $x_3 = -3$

Thus, $N(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}$

Let $\vec{y}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \vec{y}_4 = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$

<u>Gram-Schmidt Process</u>

① $\vec{x}_3 = \frac{\vec{y}_3}{\|\vec{y}_3\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

② $\vec{w}_4 = \vec{y}_4 - \langle \vec{y}_4, \vec{x}_3 \rangle \vec{x}_3 = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} - \frac{-4}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}$

$\vec{x}_4 = \frac{\vec{w}_4}{\|\vec{w}_4\|} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}$

Thus $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ forms an <u>ON</u> basis of $\mathbb{R}^4$

§6.1 $\quad A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

1. (a) Sol: $P_A(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = (\lambda^2 - 4\lambda + 3) - 8 = \lambda^2 - 4\lambda - 5$

$= (\lambda - 5)(\lambda + 1)$

$\Rightarrow \lambda_1 = 5 \quad \lambda_2 = -1$

$(A - \lambda_1 I)\vec{v}_1 = \vec{0} \iff \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \vec{v}_1 = \vec{0}$

$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \vec{v}_1 = \vec{0}$

$\iff E_A(5) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - \lambda_2 I)\vec{v}_2 = \vec{0} \iff \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \vec{v}_2 = \vec{0}$

$\iff \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \vec{v}_2 = \vec{0}$

$\iff E_A(-1) = \text{span}\{\vec{v}_2\} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

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8. Proof. Let λ be an eigenvalue of A. and $\vec{v}$ be a corresponding eigenvector then

$A\vec{v} = \lambda \vec{v} \Rightarrow A^2\vec{v} = \lambda^2 \vec{v}$

Noting that $A^2 = A$ then

$A^2\vec{v} = A\vec{v} = \lambda \vec{v}$

$\Rightarrow \lambda \vec{v} = \lambda^2 \vec{v} \Rightarrow \left.\begin{array}{l} (\lambda - \lambda^2)\vec{v} = \vec{0} \\ \vec{v} \neq 0 \end{array}\right\} \Rightarrow \lambda - \lambda^2 = 0$

$\Rightarrow \lambda = 0, 1$

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10 Sol: λ is an eigenvalue of B

$$\Leftrightarrow \quad P_B(\lambda) = 0$$
$$\Leftrightarrow \quad \det(B - \lambda I) = 0$$
$$\Leftrightarrow \quad \det(A - 2I - \lambda I) = 0$$
$$\Leftrightarrow \quad \det(A - (2+\lambda) I) = 0$$
$$\Leftrightarrow \quad P_A(2+\lambda) = 0$$
$\Leftrightarrow$ $2+\lambda$ is an eigenvalue of A. #

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§ 6.3

1. (a) $P_A(\lambda) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$

$\Rightarrow \lambda_1 = +1 \quad \lambda_2 = -1$

$(A - \lambda_1 I) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow E_A(+1) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$(A - \lambda_2 I) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow E_A(-1) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$\Rightarrow \quad X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ #

Check: $XDX^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$

2. (a) $A^6 = X D^6 X^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^6 & 0 \\ 0 & (-1)^6 \end{pmatrix} \left(\frac{1}{-2}\right) \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{(-2)} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

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3 (a) $A^{-1} = (X D X^{-1})^{-1} = X D^{-1} X^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$$

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