§1.4

3. Sol. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B$

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\boxed{\text{Answer is not unique ...}}$

9. Proof. ① $n = 4$

$$A^4 = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= O$$

② $n > 4$

$$A^n = A^4 A^{n-4} = O A^{n-4} = O$$

#

20. Proof. $(I-A)(I+A+A^2+\cdots+A^k)$

$$= (I-A)\sum_{i=0}^{k} A^i$$

$$= \sum_{i=0}^{k} A^i - A\sum_{i=0}^{k} A^i$$

$$= \sum_{i=0}^{k} A^i - \sum_{i=0}^{k} A^{i+1}$$

$$= \sum_{i=0}^{k} A^i - \sum_{i=1}^{k+1} A^i$$

$$= A^0 - A^{k+1}$$

$$= I - O$$

$$= I$$

Thus $I-A$ is nonsingular and

$(I-A)^{-1} = I + A + A^2 + \cdots + A^k$ #

§1.5

2 Sol. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

that's $E_{12}^{-1} = E_{12}$, both of which are type I elementary matrices #

(b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$\Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$. #

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix} = I_3$

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$

that is $E_{31,5}^{-1} = E_{31,-5}$, both of which are type III elementary matrices. #

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

that is, $E_{2,5}^{-1} = E_{2,\frac{1}{5}}$, both of which are type II elementary matrices #

7. sol: (a) $A = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \underset{R_2 - 3R_1}{\approx} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

$$\underset{R_1 - R_2}{\approx} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\frac{R_1}{2}}{\approx} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

That is, $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} A = I_2$

which indicates that

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

(b) Note that $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

that is, $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ #