§1.5

12 (b) $A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \Rightarrow \det A = 10 - 9 = 1 \neq 0$

$\Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

Therefore, $XA + B = C \Rightarrow XA = C - B$

$\Rightarrow X = XI_2 = XAA^{-1} = (C-B)A^{-1}$

$= \left[\begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}\right]\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

$= \begin{pmatrix} -2 & -4 \\ -8 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

$= \begin{pmatrix} 8 & -14 \\ -13 & 19 \end{pmatrix}$ #

23. Proof: A is row equivalent to B

$\Rightarrow \exists$ a finite sequence of elementary matrices, denoted as $E_1, E_2, \cdots, E_k$,

$\ni A = E_k E_{k-1} \cdots E_2 E_1 B$

$\left.\begin{array}{l} \Rightarrow B = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} A \\ \forall i,\ E_i^{-1} \text{ is elementary} \end{array}\right\}$

$\Rightarrow$ B is row equivalent to A

#

§2.1

10. Proof: (i) $n=1$. $A$ is a $2\times2$ martix with 2 identical rows

Denote $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

we have $a_{11} = a_{21}$ $a_{12} = a_{22}$

$$\Rightarrow \det A = a_{11}a_{22} - a_{12}a_{21}$$
$$= a_{11}a_{12} - a_{12}a_{11} = 0$$

(ii) Assume that the statement holds for $n=k$.

~~Let~~ $A$ be a $(k+2)\times(k+2)$ matrix with 2 identical rows

denote $A = (a_{ij})_{(k+2)\times(k+2)}$

Without loss of generality, we assume tha the 2nd & 3rd row of $A$ are identical (Switching two rows only change the sign of the determinant of the matrix)

$$\det A = \sum_{i=1}^{k+2} a_{1i} A_{1i}$$
$$= \sum_{i=1}^{k+2} (-1)^{1+i} a_{1i} \det M_{1i} \quad \cdots (*)$$

Since each minor $M_{1i}$ is a $(k+1)\times(k+1)$ matrix with its 2nd & 3rd row identical then $\det M_{1i} \equiv 0$ $i = 1, 2, \cdots, k+2$

Therefore by $(*)$ $\det A = 0$

#

§2.2

1. (a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} \underset{C_1 \leftrightarrow C_3}{=} - \begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 2 \end{vmatrix} = (-1) \times 3 \times 4 \times 2 = -24$ #

(b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} \underset{(R_4 + R_1)}{=} \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1 \times 3 \times 2 \times 5 = 30$ #

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \underset{\substack{C_1 \leftrightarrow C_2 \\ C_2 \leftrightarrow C_3 \\ C_3 \leftrightarrow C_4}}{=} - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$ #

5. Proof: denote $A = (a_{ij})_{n \times n}$ $B = \alpha A = (b_{ij})_{n \times n}$

then $b_{ij} = \alpha a_{ij}$ $\forall\ 1 \le i, j \le n$.

According to the "master formula"

$$\det(\alpha A) = \det(B) = \sum_{\substack{1 \le i_1, i_2 \cdots i_n \le n \\ \text{distinct}}} \operatorname{sign}(i_1, i_2, \cdots, i_n)\, b_{1 i_1} b_{2 i_2} \cdots b_{n i_n}$$

$$= \sum_{\substack{1 \le i_1, i_2 \cdots i_n \le n \\ \text{distinct}}} \operatorname{Sign}(i_1, i_2, \cdots i_n)\, (\alpha a_{1 i_1})(\alpha a_{2 i_2}) \cdots (\alpha a_{n i_n})$$

$$= \alpha^n \sum_{\substack{1 \le i_1, i_2 \cdots i_n \le n \\ \text{distinct}}} \operatorname{sign}(i_1, i_2, \cdots i_n)\, a_{1 i_1} a_{2 i_2} \cdots a_{n i_n}$$

$$= \alpha^n \det A$$ #