§2.2

6 Proof: A is nonsingular

$$\Rightarrow \begin{cases} \det A \neq 0 \\ \exists A^{-1} \ni A A^{-1} = I_n \Rightarrow \det A \det A^{-1} = \det(AA^{-1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad = \det(I) \\ \qquad\qquad\qquad\qquad\qquad\qquad = 1 \end{cases}$$

$$\Rightarrow \det A^{-1} = \frac{1}{\det A}. \qquad \#$$

§3.1

9 (b). Proof: If $\alpha = 0$,

$$\alpha \vec{x} = 0\vec{x} = \vec{0} \quad (\text{Thm } 3.1.1)$$

If $\alpha \neq 0$

$$\begin{aligned} \vec{x} &= 1 \cdot \vec{x} \quad (A8) \\ &= (\tfrac{1}{\alpha}\alpha)\cdot\vec{x} \\ &= \tfrac{1}{\alpha}(\alpha\vec{x}) \quad (A7) \\ &= \tfrac{1}{\alpha}\vec{0} = \vec{0} \end{aligned}$$

$$\boxed{\begin{array}{l} \forall \beta \in \mathbb{R} \quad \vec{x} \in \mathbb{R} \\ \left.\begin{array}{l} \beta\cdot\vec{x} = \beta\vec{x} + \vec{0} \quad (A3) \\ \beta\vec{x} = \beta(\vec{x} + \vec{0}) \quad (A3) \\ \quad\ \ = \beta\vec{x} + \beta\vec{0} \quad (A5) \end{array}\right\} \Rightarrow \beta\vec{0} = \vec{0} \end{array}}$$

$\#$

§3.2

11 (c) Sol: for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we have to determine if it is possible to find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

This leads to the linear system

$$\underbrace{\begin{pmatrix} -2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}}_{A} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

which always admits infinitely many sol's since

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & x \\ 1 & 3 & 4 & y \end{array}\right) \overset{R_1 \leftrightarrow R_2}{\approx} \left(\begin{array}{ccc|c} 1 & 3 & 4 & y \\ -2 & 1 & 2 & x \end{array}\right)$$

$$\overset{R_2 + 2R_1}{\approx} \left(\begin{array}{ccc|c} 1 & 3 & 4 & y \\ 0 & 7 & 10 & x+2y \end{array}\right)$$

$$\overset{R_1 - \frac{3}{7}R_2}{\approx} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{7} & -\frac{3}{7}x + \frac{y}{7} \\ 0 & 7 & 10 & x+2y \end{array}\right)$$

$$\overset{\frac{R_2}{7}}{\approx} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{7} & -\frac{3x}{7} + \frac{y}{7} \\ 0 & 1 & \frac{10}{7} & \frac{x}{7} + \frac{2y}{7} \end{array}\right)$$

that is, the 2×3 matrix A admits a RRE with 2 pivot.

Therefore $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$ is a spanning set of $\mathbb{R}^2$ #

§3.2

5(a). Sol: Let $P_1(x) = x^2$ $P_2(x) = -x^2 + x$

$P_1(x) + P_2(x) = x$ is a polynomial with odd degree.

Thus the addition is not closed.

As a result, the set of polynomials with even degrees in $\mathbb{P}_4$ is not a vector subspace #

18. Proof: $\forall$ $\vec{v} \in \mathbb{R}^{2\times 2}$, $\exists$ $a, b, c, d \in \mathbb{R}$

$$\ni \quad \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= aE_{11} + bE_{12} + cE_{21} + dE_{22}$$

Thus $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbb{R}^{2\times 2}$

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