§3.6

1 (a) Sol:

$$A=\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{pmatrix} \underset{R_3-4R_1}{\overset{R_2-2R_1}{\approx}} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus a basis for the row space

is $\{(1, 0, 2)\ (0, 1, 0)\}$

a basis for the null space

is $\left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\}$

Note that rank $A = 2$

and $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ are not linearly dependent

$\Rightarrow$ a basis for the column space

is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\}$

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6. Sol: $\vec{b}$ is in the column space

$\Rightarrow A\vec{x} = \vec{b}$ is consistent

$\Rightarrow$ there exists at least one sol; denoted as $\vec{x}_*$ such that $A\vec{x}_* = \vec{b}$ ... ①

column vectors of A are linearly dependent

$\Rightarrow A\vec{x} = \vec{0}$ admits at ~~least one~~ nontrivial sol, denoted as $\vec{x}_0$, such that

$$A\vec{x}_0 = \vec{0} \ \& \ \vec{x}_0 \neq \vec{0} \quad \text{--- ②}$$

Combining ① & ②, we have

$$A(\vec{x}_* + c\vec{x}_0) = \vec{b} \quad \text{for any } c \in \mathbb{R}.$$

Therefore $A\vec{x} = \vec{b}$ admits infinitely many sol.

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12 (a) Proof.

rank(A) = dim(Column space of A) = dim(Row space of A)

rank(B) = dim(Column space of B) = dim(Row space of B)

A & B are row equivalent ⇒ rank(A) = rank(B)

⇒ dim(Column space of A) = dim(Column space of B)

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(b) Sol. No

Counter-example $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

A & B are row-equivalent

but the column space of A, span $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

is not equal to the column space

of B, span $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

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§4.1

12 Proof: $n=2$. by definition of a Linear transformation

$$L(\alpha_1\vec{v}_1+\alpha_2\vec{v}_2)=\alpha_1 L\vec{v}_1+\alpha_2 L\vec{v}_2$$

Assume it is true for $n=k$.

then for $n=k+1$ we have

$$L(\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\cdots+\alpha_k\vec{v}_k+\alpha_{k+1}\vec{v}_{k+1})$$

$$=L(\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\cdots+1(\alpha_k\vec{v}_k+\alpha_{k+1}\vec{v}_{k+1}))$$

$$=\alpha_1 L\vec{v}_1+\alpha_2 L\vec{v}_2+\cdots+L(\alpha_k\vec{v}_k+\alpha_{k+1}\vec{v}_{k+1})$$

$$=\alpha_1 L\vec{v}_1+\alpha_2 L\vec{v}_2+\cdots+\alpha_k L\vec{v}_k+\alpha_{k+1} L\vec{v}_{k+1}$$

Therefore. by mathematical induction, the formula is true for all n.

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19 (a) $\ker(L) = \{ p \in \mathbb{P}_3 \mid x p'(x) = 0 \}$

$= \{ p \in \mathbb{P}_3 \mid p'(x) = 0 \}$

$= \operatorname{span}\{1\} = \mathbb{P}_1$

$\operatorname{Im}(L) = \{ x p'(x) \in \mathbb{P}_3 \mid p \in \mathbb{P}_3 \}$

$= \{ x(ax^2+bx+c)' \in \mathbb{P}_3 \mid a, b, c \in \mathbb{R} \}$

$= \{ 2ax^2 + bx \mid a, b \in \mathbb{R} \}$

$= \{ \hat{a}x^2 + bx \mid \hat{a}, b \in \mathbb{R} \}$

$= \operatorname{span}\{x, x^2\}$.

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