Math 421 / Homework 11.3

 # 1 For each of the following, prove that $\mathbf{f}$ and $\mathbf{g}$ are differentiable on their domains, and find formulas for $D(\mathbf{f}+\mathbf{g})(\mathbf{x})$ and $D(\mathbf{f}\cdot\mathbf{g})(\mathbf{x})$.

(a)

$$\mathbf{f}(x,y) = x - y, \quad \mathbf{g}(x,y) = x^2 + y^2.$$

(d)

$$\mathbf{f}(x,y,z) = (y, x-z), \quad \mathbf{g}(x,y,z) = (xyz, y^2).$$

 # 2(a) Find an equation of the tangent plane to $z = x^2 + y^2$ at $\mathbf{c} = (1,-1,2)$.

 # 4 Let $\mathcal{K}$ be the cone given by $z = \sqrt{x^2+y^2}$.

(a) Find an equation of each plane tangent to $\mathcal{K}$ which is perpendicular to the plane $x + z = 5$.

(b) Find an equation of each plane tangent to $\mathcal{K}$ which is parallel to the plane $x - y + z = 1$.

 # 6 Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\mathbf{a}$ and $f(\mathbf{a}) \neq 0$.

(a) Show that for $\|\mathbf{h}\|$ sufficiently small, $f(\mathbf{a}+\mathbf{h}) \neq 0$.

(b) Prove that $Df(\mathbf{a})(\mathbf{h})/\|\mathbf{h}\|$ is bounded for all $\mathbf{h} \in \mathbf{R}^n \setminus \{0\}$.

(c) If $T := -Df(\mathbf{a})/f^2(\mathbf{a})$, show that

$$\frac{1}{f(\mathbf{a}+\mathbf{h})} - \frac{1}{f(\mathbf{a})} - T(\mathbf{h}) = \frac{f(\mathbf{a}) - f(\mathbf{a}+\mathbf{h}) + Df(\mathbf{a})(\mathbf{h})}{f(\mathbf{a})f(\mathbf{a}+\mathbf{h})} + \frac{(f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}))Df(\mathbf{a})(\mathbf{h})}{f^2(\mathbf{a})f(\mathbf{a}+\mathbf{h})}$$

for $\|\mathbf{h}\|$ sufficiently small.

(d) Prove that $1/f(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{a}$ and

$$D\left(\frac{1}{f}\right)(\mathbf{a}) = -\frac{Df(\mathbf{a})}{f^2(\mathbf{a})}.$$

(e) Prove that if f and g are real-valued vector functions which are differentiable at some $\mathbf{a}$, and if $f(\mathbf{a}) \neq 0$, then

$$D\left(\frac{g}{f}\right)(\mathbf{a}) = \frac{f(\mathbf{a})Dg(\mathbf{a}) - g(\mathbf{a})Df(\mathbf{a})}{f^2(\mathbf{a})}.$$