Math 421 / Homework 5.1

 # 0 Suppose that $a < b < c$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

(a) If f is Riemann integrable on $[a, b]$, then f is continuous on $[a, b]$.

(b) If $|f|$ is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.

(d) If f is continuous on $[a, b)$ and on $[b, c]$, then f is Riemann integrable.

 # 2 (a) Prove that for each $n \in \mathbf{N}$,

$$P_n = \left\{ \frac{j}{n} \ : \ j = 0, 1, 2, \ldots, n \right\}$$

is a partition of $[0, 1]$.

(b) Prove that a bounded function f is integrable on $[0, 1]$ if

$$I_0 := \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n),$$

in which case $\int_0^1 f(x)dx = I_0$.

 # 3 Let $E = \{1/n \ : \ n \in \mathbf{N}\}$. Prove that the function

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is integrable on $[0, 1]$. What is the value of $\int_0^1 f(x)dx$?

 # 7 Let f, g be bounded on $[a, b]$.

(a) Prove that

$$(U) \int_a^b (f + g) \leq (U) \int_a^b f + (U) \int_a^b g$$

and

$$(L) \int_a^b (f + g) \geq (L) \int_a^b f + (L) \int_a^b g.$$

(b) Prove that

$$(U) \int_a^b f = (U) \int_a^c f + (U) \int_c^b f$$

and

$$(L) \int_a^b f = (L) \int_a^c f + (L) \int_c^b f$$

for $a < c < b$.

 # 8 (a) If f is increasing on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ is any partition of $[a, b]$, prove that

$$\sum_{j=1}^{n} (M_j(f) - m_j(f))\Delta x_j \leq (f(b) - f(a))\|P\|.$$

(b) Prove that if f is monotone on $[a, b]$, then f is integrable on $[a, b]$.