**Math 421 / Homework 8.3**

**# 1** Sketch each of the following sets. Identify which of the following sets are open, which are closed, and which are neither. Also discuss the connectivity of each set.

(b) $E = \{(x, y): \ x^2 + 4y^2 \le 1\}$

(c) $E = \{(x, y): \ y \ge x^2, \ 0 \le y < 1\}$

(d) $E = \{(x, y): \ x^2 - y^2 > 1, \ -1 < y < 1\}$

(e) $E = \{(x, y): \ x^2 - 2x + y^2 = 0\} \cup \{(x, 0): \ x \in [2, 3]\}$

**# 2** Let $n \in \mathbf{N}$, let $\mathbf{a} \in \mathbf{R}^n$, let $s, r \in \mathbf{R}$ with $0 < s < r$, and set

$$V = \{\mathbf{x} \in \mathbf{R}^n : s < \|\mathbf{x} - \mathbf{a}\| < r\}, \quad E = \{\mathbf{x} \in \mathbf{R}^n : s \le \|\mathbf{x} - \mathbf{a}\| \le r\}.$$

Prove that $V$ is open and $E$ is closed.

**# 5** (a) Let $E_1$ denote the closed ball centered at $(0, 0)$ of radius 1 and $E_2 = B_{\sqrt{2}}(2, 0)$, and sketch a graph of the set

$$U := \{(x, y): \ x^2 + y^2 \le 1 \ \text{and} \ x^2 - 4x + y^2 + 2 < 0\}.$$

(b) Decide whether $U$ is relatively open or relatively closed in $E_1$. Explain your answer.

(c) Decide whether $U$ is relatively open or relatively closed in $E_2$. Explain your answer.

**# 7** (b) If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected sets in $\mathbf{R}^n$ and $\cap_{\alpha \in A} E_\alpha \ne \emptyset$, prove that

$$E = \cup_{\alpha \in A} E_\alpha$$

is connected.

(d) Find two connected sets $A$ and $B$ in $\mathbf{R}^2$ with $A \cap B \ne \emptyset$, but $A \cap B$ is not connected.

**# 9** Show that if $E$ is closed in $\mathbf{R}^n$ and $\mathbf{a} \notin E$, then

$$\inf_{\mathbf{x} \in E} \|\mathbf{x} - \mathbf{a}\| > 0.$$