Math 421 / Homework 9.2

 # 5 Let E be closed and bounded in $\mathbf{R}$, and suppose that for each $x \in E$ there is a function f_x, nonnegative, nonconstant, increasing, and C^∞ on $\mathbf{R}$, such that $f_x(x) > 0$ and $f_x'(y) = 0$ for $y \notin E$. Prove that there exists a nonnegative, nonconstant, increasing C^∞ function f on $\mathbf{R}$ such that $f(y) > 0$ for all $y \in E$ and $f'(y) = 0$ for all $y \notin E$.

 # 6 (Note that this problem is not exactly the one in the text; the assumption that K is compact is dropped.) Suppose that $\mathbf{f}\colon \mathbf{R}^n \to \mathbf{R}^m$ and that $\mathbf{a} \in K$, where K is a connected subset of $\mathbf{R}^n$. Suppose further that for each $\mathbf{x} \in K$ there exists a $\delta_{\mathbf{x}} > 0$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y})$ for all $\mathbf{y} \in B_{\delta_{\mathbf{x}}}(\mathbf{x})$. Prove that $\mathbf{f}$ is constant on K; that is, $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$ for all $\mathbf{x} \in K$.

 # 7 Define the distance between two nonempty subsets A, B of $\mathbf{R}^n$ by

$$\operatorname{dist}(A, B) := \inf\{\|\mathbf{x} - \mathbf{y}\| : \ \mathbf{x} \in A, \ \mathbf{y} \in B\}.$$

(a) Prove that if A and B are compact sets which satisfy $A \cap B = \emptyset$, then $\operatorname{dist}(A, B) > 0$.

(b) Show that there exist nonempty, closed sets A, B in $\mathbf{R}^2$ such that $A \cap B = \emptyset$ but $\operatorname{dist}(A, B) = 0$.