Name: Answer Key

Math 132: Section 21

Midterm 2

October 19, 2009

The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 100 possible points. Good luck!

1	
2	
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6	
Total	

1. (20 points) a. If $y = x^4 - 3x^2$, then $\frac{dy}{dx} = ?$

$$\boxed{\frac{dy}{dx} = 4x^3 - 6x}$$

b. Find $\frac{dy}{dx}$ given that $x = \dfrac{y^2+1}{y-x}$.

$$x = \frac{y^2+1}{y-x}$$

$$x(y-x) = y^2+1$$

$$xy - x^2 = y^2+1$$

$$\frac{d}{dx}(xy - x^2) = \frac{d}{dx}(y^2+1)$$

$$y + x\frac{dy}{dx} - 2x = 2y\, dy/dx$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx}(x - 2y) = 2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{2x-y}{x-2y}}$$

or

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\frac{y^2+1}{y-x}\right)$$

$$1 = \frac{(y-x)(2y\, dy/dx) - (y^2+1)\left(\frac{dy}{dx} - 1\right)}{(y-x)^2}$$

$$(y-x)^2 = (2y^2 - 2xy)\frac{dy}{dx} - (y^2+1)\frac{dy}{dx} + (y^2+1)$$

$$(y-x)^2 - (y^2+1) = (2y^2 - 2xy - y^2 - 1)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{(y-x)^2 - (y^2+1)}{2y^2 - 2xy - y^2 - 1}}$$

$$= \frac{\cancel{y^2} - 2xy + x^2 - \cancel{y^2} - 1}{y^2 - 2xy - 1}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 2xy - 1}{y^2 - 2xy - 1}}$$

Note: These can be seen to be the same using substitution $y^2+1 = x(y-x)$

2. (20 points) a. Suppose $x(t) = \sin(\sin(t^3))$. Find $\frac{dx}{dt}$.

$$\frac{dx}{dt} = \cos(\sin t^3)\, \frac{d}{dt}(\sin t^3)$$

$$= \cos(\sin t^3) \cos t^3\, \frac{d}{dt}(t^3)$$

$$\boxed{= \cos(\sin(t^3)) \cos(t^3)\, 3t^2}$$

b. When $g(x) = \sin(x^2)$, find $\frac{d^2g}{dx^2}$ at $x = \sqrt{\frac{\pi}{4}}$.

$$\frac{dg}{dx} = \cos(x^2) \cdot 2x$$

$$\frac{d^2g}{dx^2} = \frac{d}{dx}[\cos x^2]\, 2x + \cos x^2\, \frac{d}{dx}[2x]$$

$$= -\sin(x^2) \cdot 2x \cdot 2x + \cos(x^2) \cdot 2$$

$$= -4x^2 \sin(x^2) + 2\cos(x^2)$$

$$\left.\frac{d^2g}{dx^2}\right|_{x=\sqrt{\pi/4}} = -4\left(\sqrt{\pi/4}\right)^2 \sin\left(\sqrt{\pi/4}^{\,2}\right) + 2\cos\left(\sqrt{\pi/4}^{\,2}\right)$$

$$= -4 \cdot \frac{\pi}{4} \cdot \sin(\pi/4) + 2\cos(\pi/4)$$

$$= -\pi \frac{\sqrt{2}}{2} + \cancel{2}\frac{\sqrt{2}}{\cancel{2}}$$

$$\boxed{= \frac{-\pi\sqrt{2}}{2} + \sqrt{2}}$$

3. (10 points) Find $\frac{d}{dx}(3x^2)$ using the definition of the limit.

$$\frac{d}{dx}(3x^2) = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h^2}$$

$$= \lim_{h \to 0} \frac{3(x^2+2xh+h^2) - 3x^2}{h^2}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h^2}$$

$$= \lim_{h \to 0} \frac{h}{h}(6x+3h)$$

$$= \lim_{h \to 0} 6x + 3h = \boxed{6x}$$

4. (15 points) Write the equation for the tangent line to the curve $y = x^5$ at the point $x = 2$.

$f(x) = x^5$ $\qquad$ $f(2) = 2^5 = 32$

$f'(x) = 5x^4$ $\qquad$ $f'(2) = 5 \cdot 2^4 = 80$

Tangent line goes through point $(2, f(2)) = (2, 32)$
w/ slope 80

$\boxed{y - 32 = 80(x-2)}$

$\boxed{y = 80(x-2) + 32}$

b. Use linear approximation to estimate $(2.001)^5$.

$L(x) = 80(x-2) + 32$

$(2.001)^5 = f(2.001) \approx L(2.001) = 80(2.001-2) + 32$

$= 80(.001) + 32$

$= .08 + 32$

$= 32.08$

$\boxed{(2.001)^5 \approx 32.08}$

5. (15 points) If a ball is thrown vertically upward with an initial velocity of 64 ft/sec, then its height in feet after t seconds is

$$s(t) = 64t - 16t^2.$$

What is the maximum height reached by the ball?

Max ht occurs when $v(t) = 0$

$v(t) = s'(t) = 64 - 32t$

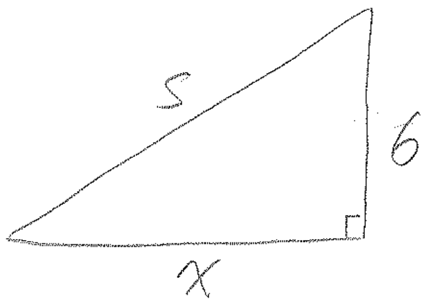
$0 = v(t) = 64 - 32t$

$t = \frac{64}{32} = 2$ } Time when max height occurs

$s(2) = 64(2) - 16(2^2) = 2 \cdot 64 - \cancel{3}4 = 64$

$\boxed{\text{Max ht is } s(2) = 64 \text{ ft}}$

6. (20 points) A boat is pulled toward a dock by a rope from the bow (tip of the boat) through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec. What is the boat's speed when 10 ft of rope are out?



$\frac{ds}{dt} = -2$

What is $\frac{dx}{dt}$ when $s = 10$?

$x^2 + 6^2 = s^2$

$\frac{d}{dt}(x^2+6^2) = \frac{d}{dt}(s^2)$

$2x \frac{dx}{dt} = 2s \frac{ds}{dt}$

$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$

$\rightarrow$ If $s = 10$,

$x^2 = 10^2 - 6^2 = 100 - 36$

$x^2 = 64$

$\underline{x = 8}$

when $s=10$ $\quad \frac{dx}{dt} = \frac{10}{8}(-2) = \frac{-10}{5} = \frac{-5}{2} = -2.5$

$\boxed{\text{The speed of the boat is then } 2.5 \text{ ft/sec when } s = 10}$