NOTES FOR 02/28/2011, 03/02/2011

MATH 309, SECTION 6

Note: We will not cover Chapter 4 in this class. Chapter 5 will be covered, but after covering some of Chapter 6.

1. Coordinates (Chapter 3.6)

From our earlier work, we know the following important theorem concerning bases for vector spaces.

Theorem 1. *Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for the vector space V. Then, any vector $\mathbf{v} \in V$ can be expressed uniquely as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$. In other words, there is a unique solution $(r_1, \dots, r_n)$ to the equation*

$$r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n = \mathbf{v}.$$

Definition 1. Suppose $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for the vector space V. Then, for a vector $\mathbf{v} \in V$, we say that the **coordinates** (or coordinate vector) of $\mathbf{v}$ with respect to the basis B is the unique vector $(r_1, \dots, r_n) \in \mathbb{R}^n$ such that $\mathbf{v} = \sum_i r_i\mathbf{v}_i$. We use the notation

$$[\mathbf{v}]_B = [r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n]_B = (r_1, \dots, r_n) = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}.$$

Example 2. $\mathbb{R}^n$ has the canonical basis $B = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ where $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$, and

$$[(x_1, \dots, x_n)]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Example 3. The vector space $\mathbb{P}_n$ has a standard basis $\{1, x, \dots, x^n\}$, and

$$[\sum_i a_i x^i]_B = (a_0, \dots, a_n).$$

Example 4. Let $B = \{(1, 2), (3, 1)\}$ be a basis for $\mathbb{R}^2$. Then, to find the coordinates of an arbitary vector $(a, b) \in \mathbb{R}^2$ with respect to B, we solve the equation

$$r_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + r_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 3 & a \\ 2 & 1 & b \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -\frac{1}{5}a + \frac{3}{5}b \\ 0 & 1 & \frac{2}{5}a - \frac{1}{5}b \end{bmatrix}$$

Therefore,

$$[(a, b)]_B = \begin{bmatrix} -\frac{1}{5}a + \frac{3}{5}b \\ \frac{2}{5}a - \frac{1}{5}b \end{bmatrix}.$$

More concretely,

$$[(5, 5)]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Note: the *order* of the vectors in the basis matters! Swapping the order will swap the corresponding columns in the coordinate vector.

Example 5. Consider the subspace V of $\mathbb{M}(2,2)$ with the basis

$$B = \left\{ \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Then, the coordinate vector $(5, -2) \in \mathbb{R}^2$ represents the matrix

$$5 \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -7 & -2 \\ 8 & 0 \end{bmatrix}$$

relative to the basis B.

To find the coordinates of $\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$ relative to B, we solve

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and conclude that the coordinate vector is $(-2, 1) \in \mathbb{R}^2$.

2. Linear maps (Chapter 6.1)

The previous examples are all examples of *maps between vector spaces.* Given a finite-dimensional vector space V with basis B, we have a function (or mapping) that associates to any vector $\mathbf{v} \in V$ a vector in $\mathbb{R}^n$:

$$\begin{aligned} \mathbb{R}^n &\xleftarrow{[]_B} V \\ [\mathbf{v}]_B &\longleftarrow \mathbf{v} \end{aligned}$$

Remark 6. The book (and probably all of your previous textbooks) would usually write the above as $[]_B : V \to \mathbb{R}^n$, which is read left to right. We will use the "right to left" notation. While it is a little confusing at first, it will be much more convenient later in the course when encountering function composition and matrix multiplication.

Definition 7. Let V and W be vector spaces. A **function** T from V to W, written $T : V \to W$, is a rule that assigns to each vector $v \in V$ a unique vector $T(v) \in W$.

Vocabulary: The textbook uses the word *function*, and the words *transformation* and *map* or *mapping* are also common; all have the same meaning. Given a function $W \xleftarrow{T} V$,

- V is called the *domain* and W is the *target space*.
- If $\mathbf{w} = T(\mathbf{v})$, then $\mathbf{w}$ is the *image of* $\mathbf{v}$ under T.
- The set of all images is called the *range* of T. The range may be a part of W or all of W.

Example 8. The function $f(x) = x^2$ has domain and target space $\mathbb{R}$.

A curve in the plane is a function $\mathbb{R}^2 \leftarrow \mathbb{R}$, and a curve in $\mathbb{R}^3$ is a function $\mathbb{R}^3 \leftarrow \mathbb{R}$. The domain is $\mathbb{R}$ in both cases, and the target space is $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

A vector field on the plane is a map $\mathbb{R}^2 \leftarrow \mathbb{R}^2$. The domain and target space are both $\mathbb{R}^2$.

Note that none of the above examples are assumed to be linear. The notions of domain/range/target apply to functions in general and do not rely on vector space structures.

Definition 9. A function $W \xleftarrow{T} V$ between vector spaces is **linear** if for all $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$,

$$T(r\mathbf{v}) = r\,T(\mathbf{u}) \qquad \text{and} \qquad T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}).$$

Lemma 1. *If $W \xleftarrow{T} V$ is linear, then for all $\mathbf{u}, \mathbf{v} \in V$ and $a, b \in \mathbb{R}$:*

$$(a)\ \ T(\mathbf{0}) = \mathbf{0} \qquad (b)\ \ T(-\mathbf{v}) = -T(\mathbf{v}) \qquad (c)\ \ T(a\mathbf{u} + b\mathbf{v}) = a\,T(\mathbf{u}) + b\,T(\mathbf{v}).$$

and (c) extends to general linear combinations: $T\left(\sum a_i \mathbf{v}_i\right) = \sum a_i\, T(\mathbf{v}_i)$.

Proof.

$$\begin{aligned} T(\mathbf{0}_V) &= T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}_W, \\ T(-\mathbf{v}) &= T((-1)\mathbf{v})) = (-1)T(\mathbf{v}) = -T(\mathbf{v}), \\ T(a\mathbf{u} + b\mathbf{v}) &= T(a\mathbf{u}) + T(b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}). \end{aligned}$$

□

Remark 10. The above lemma shows that T linear implies $T(r\mathbf{u} + s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$. The converse is also true, as demonstrated by setting $r = 1, s = 1$ or $s = 0$. Therefore, being linear is equivalent to

$$T(r\mathbf{u} + s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$$

being satisfied for all $r, s \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$.

Example 11. The derivative is a linear map $\mathbb{C}(\mathbb{R}) \overset{\frac{d}{dx}}{\leftarrow} \mathbb{D}(\mathbb{R})$. This follows from standard properties of derivatives, as

$$\frac{d}{dx}(rf + sg) = \frac{d}{dx}(rf) + \frac{d}{dx}(sg) = r\frac{df}{dx} + s\frac{dg}{dx}.$$

Example 12. The linear map $\mathbb{P}_3 \overset{T}{\leftarrow} \mathbb{P}_2$ given by $T(p) = (x + 1)p$ is linear. Check:

$$\begin{aligned} T(rp_1 + sp_2) &= (x + 1)(rp_1 + sp_2) = r(x + 1)p_1 + s(x + 1)p_2 \\ &= rT(p_1) + sT(p_2). \end{aligned}$$

Example 13. Given a basis B of V, the "coordinates" are really a linear map $\mathbb{R}^n \leftarrow V$. Checking this is linear is a homework assignment.

Lemma 2. *Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for the vector space V. A linear transformation $W \overset{T}{\leftarrow} V$ is determined by the values $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$; i.e.*

(a) *If we know $T(\mathbf{v}_i)$ for all i, we can calculate $T(\mathbf{v})$ for any vector $\mathbf{v} \in V$.*

(b) *If $W \overset{S}{\leftarrow} V$ is a linear map so that $S(\mathbf{v}_i) = T(\mathbf{v}_i)$ on each basis vector $\mathbf{v}_i$, then $S(\mathbf{v}) = T(\mathbf{v})$ for all vectors $\mathbf{v}$ in V.*

Proof. Given a basis B of V, any vector $\mathbf{v} \in V$ is uniquely written as $\mathbf{v} = \sum_i r_i\mathbf{v}_i$. If T is a linear map, then

$$T(\mathbf{v}) = T(\sum_i r_i\mathbf{v}_i) = \sum_i r_iT(\mathbf{v}_i),$$

so T is completely determined by its values on the basis vectors. Similarly, if S is another linear map which agrees with T on basis vectors, then

$$S(\mathbf{v}) = S(\sum_i r_i\mathbf{v}_i) = \sum_i r_iS(\mathbf{v}_i) = \sum_i r_iT(\mathbf{v}_i) = T(\mathbf{v}).$$

□