Elementary Row Operations

1. Interchange two rows (e.g. ($R1 \leftrightarrow R2$).
2. Multiply one row by a nonzero number (e.g. $R2 \times 3$).
3. Add a multiple of one row to a different row (e.g. $R2 - 2R3$).

Elementary row operations do not change the solution set. Thus we can solve a system of linear equations by the following *Gauss-Jordan Elimination procedure*:

1. Write the system as an augmented matrix.
2. Apply elementary row operations to transform the coefficient matrix into an identity matrix (if possible):
   (a) Interchange rows, if necessary, to make the top left entry ("first pivot") non-zero.
   (b) Divide Row 1 by the pivot.
   (c) Subtract multiples of Row 1 to achieve 0s beneath the first pivot.
   (d) Interchange Row 2 with a later rows, if necessary, to make the the second entry of Row 2 ("second pivot") non-zero.
   (e) Divide Row 2 by its pivot.
   (f) Subtract multiples of Row 2 to achieve 0s above and beneath the second pivot, etc.
3. Read off the solution and *check that your answer is a solution.*

Example 1 Use elementary row operations to solve the system

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x \quad\; - z &= 3 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{bmatrix} \overset{R2-2R1}{\approx} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 3 & 0 & -1 & 3 \end{bmatrix} \overset{R3-3R1}{\approx} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{bmatrix} \overset{R2\div(-5)}{\approx}$$

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{bmatrix} \overset{R1-2R2}{\approx} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{bmatrix} \overset{R3+6R2}{\approx} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{bmatrix} \overset{R3\div(-4)}{\approx}$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \overset{R1-R3}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \overset{R2-R3}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore x = 2$
$y = -1$
$z = 3.$
Does it work?✓

Homework 1

1. State the size of the following matrices (e.g. 2×3):

$$(a) \quad \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 9 \\ 4 & 8 \end{bmatrix} \qquad (b) \quad \begin{bmatrix} 6 & 2 & -2 & 4 & 7 \\ 0 & 1 & -3 & 12 & 5 \\ 6 & -1 & 8 & 2 & -9 \end{bmatrix} \qquad (c) \quad \begin{bmatrix} 5 & -6 & 3 & 1 \end{bmatrix}$$

2. Write down the identity matrix I_4.

3. Write the coefficient matrix and the augmented matrix for the following linear systems.

$$(a) \quad \begin{aligned} x_1 - 2x_2 &= -8 \\ 2x_1 - 3x_2 &= -11 \end{aligned} \qquad (b) \quad \begin{aligned} x + y + 3z &= 6 \\ x + 2y + 4z &= 9 \\ 2x + y + 6z &= 11 \end{aligned}$$

4. Use Gauss-Jordan Elimination to solve each of the linear systems in Problem 3. Follow the format of Example 1 on the back of this page.

5. Use Gauss-Jordan Elimination (with the same format) to solve the following two linear systems.

$$(c) \quad \begin{aligned} x_1 - x_2 - x_3 &= -1 \\ -2x_1 + 6x_2 + 10x_3 &= 14 \\ 2x_1 + x_2 + 6x_3 &= 9 \end{aligned} \qquad (d) \quad \begin{aligned} x_1 + 2x_2 + 2x_3 + 5x_4 &= 11 \\ 2x_1 + 4x_2 + 2x_3 + 8x_4 &= 14 \\ x_1 + 3x_2 + 4x_3 + 8x_4 &= 19 \\ x_1 - x_2 + x_3 \qquad &= 2 \end{aligned}$$

6. Find a quadratic function $f(x) = ax^2 + bx + c$ whose graph passes through the points $(1, 4), (3, 18)$ and $(-2, 28)$. (*Begin by writing the conditions* $f(1) = 4$, $f(3) = 18$ *and* $f(-2) = 28$ *as a linear system for the unknowns* a, b, c.)

Problems and Notes

Row echelon form, reduced row echelon form, or neither?

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 2 & 9 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Gaussian Elimination Algorithm (also called Gauss-Jordan elimination or Gauss-Jordan reduction)

1. Interchange rows, if necessary, to bring a non-zero number (the **pivot**) to the top of the first non-zero column.
2. Create a 1 at the pivot position by multiplying the first row by $\frac{1}{\text{pivot}}$.
3. Create zeros elsewhere in the pivot column by adding suitable multiples of the pivot row to the other rows.
4. Cover the pivot row and all rows above it. Repeat Steps 1 and 2 for the remaining submatrix. Repeat Step 3 for the whole matrix.

Continue until reduced row echelon form is reached.

Do next step in Gaussian elimination:

$$\begin{bmatrix} 1 & 5 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & -5 & 1 & 0 \end{bmatrix} \approx$$

$$\begin{bmatrix} 1 & -1 & -\frac{4}{7} & 0 & 6 \\ 0 & 1 & \frac{4}{7} & 0 & -6 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{9}{8} \end{bmatrix} \approx$$

Overdetermined systems (more equations than unknowns) are usually inconsistent (no solutions). There are no solutions whenever the augmented matrix, in reduced row echelon form, has a row of the form $(0, 0, \ldots, 0, a)$ with $a \neq 0$.

Underdetermined systems (fewer equations than unknowns) usually have infinitely many solutions. To write the solution set, introduce one free variable $r, s, t \cdots \in \mathbb{R}$ for each non-pivot column and express the general solution in terms of these.

For example, if the reduced augmented matrix is

$$\begin{bmatrix} 1 & 5 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 7 \end{bmatrix} \quad \text{then the system is} \quad \begin{aligned} x_1 + 2x_2 - x_5 &= 3 \\ x_3 + 4x_5 &= 5 \\ x_4 + x_5 &= 7 \end{aligned}$$

The non-pivot columns are the second (x_2) and the fifth (x_5), so write $x_2 = r$ and $x_5 = s$. The equations are then $x_1 = 3 - 2r + s, x_3 = 5 - 4s$ and $x_4 = 7 - s$, so the solutions set is the set of all 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ satisfying these equations, which we write as

$$S = \Big\{(3 - 2r + s, r, 5 - 4s, 7 - s, s) \;\Big|\; r, s \in \mathbb{R}\Big\}.$$

Homework 2

Do the following problems in the textbook (some answers in the back of the book):

1. Page 75, Problems 5a and 5c (on back of this page).
2. Page 75, Problems 6a and 6d.
3. Page 75, Problem 9.
4. Page 79, Problems 5a, 5b and 5d.
5. Page 79, Problem 7b.
6. There is a unique circle through any three non-colinear points. The general equation for a circle in the xy-plane is $x^2 + y^2 + ax + by + c = 0$.
   (a) Find the circle that passes through the points $(0, 1), (-1, 4)$, and $(2, 1)$.
   (b) Check your answer by completing the square, finding the center and radius of the circle, and sketching the circle.