Math 309 Notes, courtesy Prof. Parker

Notes on logic

statements

In mathematics, a *statement* is an assertion that can be classified as either true or false within the mathematical context that it appears. For example,

- $\sqrt{2}$ is less than 3.
- There is no x such that $x^2 = -1$.

Note that each word in a statement must have a clearly-defined meaning; “Problem 1 is too hard” is an assertion, but is not a statement unless we have a precise definition of “too hard”.

∃ and ∀

When making statements, it is important to be clear whether you are asserting that *every* object of a certain type satisfies a condition, or whether you aere simply saying that there is one that does. This is done by using one of two quantifiers:

- *There exists*, written ∃, is used to say that there is at least one such object.

  Example: “∃ a real number less that 0” .

- *For all*, written ∀, is used to say that a property holds for all objects of a certain type.

  Example: “‘$x^2 \geq 0$ holds $\forall x \in \mathbb{R}$” .

It’s easy to prove or verify a “there exists” statement: just produce one example. Proving or verifying a “for all” statement is harder: you must give an argument that the statement is true for *every* example.

∃!

The word *unique* is often used together with “there exists”. The phrase “there exists a unique”, written ∃!, is used to say that there is *exactly one* such object.

Example: “‘for each positive real number, ∃! $y \in \mathbb{R}$ such that $y^2 = x$” .

To prove or verify such a statement involves two steps: (i) produce one example, and (ii) argue that there are no other examples.

and or

One can combine two or more statements into a compound statement using the words *and* and *or*. The word “and” simply means that both statements are true. In mathematics (but not in ordinary language) the word “or” is always used in a nonexclusive way: a statement of the form “A or B” is true if A, or B *or both* A and B are true.

Examples: “$\sqrt{2} < 5$ and $\sqrt{2} > 0$” is TRUE; “$\frac{1}{3} < 1$ or $0 < 1$” is TRUE.

negation

Every statement has a negation; the negation is the statement that asserts the opposite of the original statement. One can negate a statement by saying “It is not true that …”. But this is not helpful. One should try to cast the negation as a direct statement. It is important to realize that

- negations reverse the words *and* and *or*.
- negations reverse the quantifiers *there exists* and *for all*.

conditional statement, theorem

A *conditional statement* is a statement of the form “If A then B” for some statements A and B. Statement A is called the *hypothesis* and statement B is called the *conclusion*. A *theorem* is a conditional statement that has been proved to be true. Note that a conditional statement may be either true or false, but it is not called a theorem until it has been proved. Also note that the theorem does not mean that B is true in all cases — only that it is true when the hypothesis holds. *Every theorem can be stated in if-then form.*

implies

The statement $A \implies B$ ("A implies B) means that B is true in every case in which A is true..

Example: "If $x > 1$ then $2x > 2$" is TRUE – it is a theorem. But the hypothesis is not true for all x.

Beginning of class exercise

1. Negate each of the following statements.
   (a) Every triangle has an angle sum of 180°.
   (b) It is cold and rainy outside.
   (c) My favorite color is red or green.
   (d) If the sun shines then we play tennis.
   (e) All students wear green on St. Patrick's day.
2. Rephrase using $\exists$, $\forall$ and $\exists!$:
   (a) Every polynomial $p(x)$ of odd degree has a root.
   (b) Not every real number has a square root.
   (c) There is only one number whose tenth power is 0.
3. Identify the hypothesis and the conclusion in each statement and rewrite as an "If – then" statement.
   (a) I can take topology if I pass linear algebra.
   (b) All even numbers are divisible by 4.
   (c) A number is divisible by 4 only if it is even.

Axiomatic systems

Most areas of modern mathematics are organized axiomatically. This is the approach that Euclid took to developing geometry: he began with 5 axioms (Euclid's "postulates") and built the entire subject from there. In the 19th and early 20th centuries this scheme was applied to other areas of mathematics and refined into what is known as the *axiomatic system*. In particular, linear algebra is organized as an axiomatic system.

There are four parts of an axiomatic system.

1. Terms and definitions. Each axiomatic system begins with a few *undefined terms*. The undefined terms in linear algebra are *point*, *vector*, *set*, *element*, and a few others. After that, new terms are introduced with precise definitions.

2. Axioms. An *axiom* or *postulate* (the words are interchangeable) is a statement that is accepted without proof. The subject (linear algebra for us) begins with a short list of axioms. Everything else is logically derived from them.

Sometimes axioms refer to other mathematical subjects or objects. The axioms of linear algebra assume that you are familiar with the properties of the real numbers, and of sets (which themselves can be developed as axiomatic systems).

3. Theorems. The largest part, by far, of an axiomatic system consists of theorems and their proofs. A *theorem* is a "if–then" statement that has been proved to be a logical consequence of the axioms. Vocabulary: the words *theorem* and *proposition* are synonyms, a *lemma* is a theorem that is stated as a step toward some more important result, and a *corollary* is a theorem that can be quickly and esaily deduced from a previously-proved theorem.

Theorems are organized in a strict logical order, building on each other. The first theorem is proved using only the axioms. The second theorem is proved using only the first theorem and the axioms, etc..

4. Models. An axiomatic system is a purely logical construction. The human brain is not good at understanding complicated logical constructions. Thus it is advantageous to have a way of thinking about or visualizing the undefined terms that is compatible with human experience.

An *interpretation* of an axiomatic system is a particular way of giving meaning to the undefined terms in that system. An interpretation is called a *model* if the axioms are true statements in that interpretation. As a result, all of the theorems are also true for the model.

A good model makes it possible to visualize and guess theorems, and it provides guidance in developing proofs.

Axioms of a vector space

A *vector space* is a set V (whose elements are called vectors) endowed with

- a rule for addition that associates to each pair $\mathbf{v}, \mathbf{w} \in V$ an element $\mathbf{v} + \mathbf{w} \in V$, and
- a rule for scalar multiplication that associates to each $\mathbf{v} \in V$ and $r \in \mathbb{R}$ an element $r\mathbf{v} \in V$,

such that, for all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$ and $r, s \in \mathbb{R}$,

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	*addC*	Addition is commutative
2. $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{w} + (\mathbf{v} + \mathbf{x})$	*addA*	Addition is associative
3. $\exists$ a vector $\mathbf{0} \in V$ s.t. $\mathbf{v} + \mathbf{0} = \mathbf{v}$	*addId*	Additive Identity Property
4. For each $\mathbf{v} \in V$, $\exists$ an "opposite vector" $-\mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	*addInv*	Additive Inverse Property
5. $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$ and $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	*distrib*	Distributive Property
6. $r(s\mathbf{v}) = (rs)\mathbf{v}$	*multA*	Multiplication is associative
7. $1 \cdot \mathbf{v} = \mathbf{v}$	*multId*	Multiplication Identity Property

Notes (a) These axioms implicitly assume that the properties of the real numbers, of sets, and of the symbol = (e.g. adding the same thing to both sides preserves equality) are known and will be used freely. Thus is proofs you will have occasion to use the abbreviations

prop of $\mathbb{R}$ *prop of sets* *prop of* =

for "properties of the real numbers" , "properties of sets" and "properties of equality".

(b) In the context of Axioms 3 and 7, the word "identity" means "vector that makes no change".

(c) When checking it a given set is a vector space, the first bulleted requirements are the most important.

From the axioms, one can derive numerous simple consequences that are useful in calculations. Each is proved from the axioms *and from previously proved facts.* Once a fact is proved, it gets added to our basket of "known facts" and can be used from then on.

Examples of Vector Spaces

1. **The real numbers** $\mathbb{R}$ with the usual addition and scalar multiplication.

2. **The vector spaces $\mathbb{R}^n$, Euclidean space with coordinate grid**, is the set of n-tuples or real numbers (written either vertically or horizontally). For $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, the sum is defined by adding corresponding components, and the scalar multiplication by $r \in \mathbb{R}$ is defined by multiplying every component by r:

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \qquad \text{and} \qquad r\mathbf{v} = (rv_1, rv_2, \ldots, rv_n)$$

3. **The vector space $\mathbf{M}(m, n)$ of all $m \times n$ matrices.** The sum of matrices A and B is defined as the $m \times n$ matrix whose ij entry is the sum of the ij entry of A and the ij entry of B. Scalar multiplication by $r \in \mathbb{R}$ is defined multiplying each entry in A by r.

4. **The vector space P_n of polynomials of degree less than** n consists of all polynomials p of the form

$$p(x) \;=\; a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

in the variable x. Note that some, or even all, of the coefficients can be 0. Addition and scalar multiplication are done on each coefficient separately: if $q(x) \;=\; b_0 + b_1 x + \cdots + b_n x^n$ then

$$(p+q)(x) \;=\; (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n \qquad (rp)(x) \;=\; ra_0+(ra_1)x+\cdots+(ra_n)x^n,$$

This is the same as thinking of p and q as functions of x and adding and scalar multiplying as in the next example.

5. **The vector spaces $\mathcal{C}[a, b]$ of continuous functions.** Let $\mathcal{C}[a, b]$ denote the set of all real-valued functions defined and continuous on the interval $a \leq x \leq b$. The sum of functions $f + g$ is defined by taking the sum of the real numbers $f(x)$ and $f(x)$ at each x: of gegree less than n. Define $p + q$ and rp by

$$(f + g)(x) \;=\; f(x) \;+\; g(x)$$

This is an element in $\mathcal{C}[a, b]$ because the sum of continuous function is continuous. Scalar multiplication by $r \in \mathbb{R}$ is defined similarly:

$$(rf)(x) \;=\; r \cdot f(x).$$

Subspaces (cf. Section 1.8 of the textbook.)

Definition. If subset S of a vector space V is a *vector subspace* if it has the following properties:

(i) S is not empty.

(ii) S is *closed under addition:* If $\mathbf{v}$ and $\mathbf{w}$ are both in S, then so is $\mathbf{v}+\mathbf{w}$.

(iii) S is *closed under scalar multiplication:* If $\mathbf{v} \in S$ then $r\mathbf{v}$ is in S for every scalar $r \in \mathbb{R}$.

Remarks. 1. In a vector space V, one can readily check that $\{\mathbf{0}\}$ and V are vector subspaces. All other subspaces are called *proper subspaces.* We refer to $\{\mathbf{0}\}$ as the *zero subspace.*

2. Every subspace must contain the zero vector; if it does then it is non-empty. Thus a simple way to verify that a given subset is a subspace is to

(a) Show that $\mathbf{0} \in S$, and

(b) Prove that $\mathbf{v}, \mathbf{w} \in S \implies \mathbf{v}+\mathbf{w} \in S$.

(c) Prove that $\mathbf{v} \in S \implies r\mathbf{v} \in S \quad \forall r \in \mathbb{R}$.

Alternatively, one can combine the two types of closure by checking closure under linear combinations, replacing steps (b) and (c) by

(bc) Prove that if $\mathbf{v}, \mathbf{w} \in S$ then $r\mathbf{v} + s\mathbf{w} \in S$ for all $r, s \in \mathbb{R}$.

Example. The set $S = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ is a subspace of $\mathbb{R}^2$.

Proof: (a) $\mathbf{0} = (0, 0) \in S$ ✓ (i.e. the zero vector of $\mathbb{R}^2$ satisfies $x = y$).

If $\mathbf{v} = (x_1, y_1)$ and $\mathbf{w} = (x_2, y_2)$ are in S then $x_1 = y_1$ and $x_2 = y_2$. Then

(b) $\mathbf{v} + \mathbf{w} = (x_1 + x_2, y_1 + y_2) \in S$ because $x_1 + x_2 = y_1 + y_2$.

(c) For any $r \in \mathbb{R}$, $r\mathbf{v} = (rx_1, ry_1) \in S$ because $rx_1 = ry_1$.

Thus S is a subspace of $\mathbb{R}^2$. □

Subspace Theorem. A subspace S of a vector space V is itself a vector space (using the addition and scalar multiplication of V).

Proof: Done in class and in the textbook page 40. □

Intersection Theorem. The intersection of subspaces is a subspace.

Proof: Let $V_1, \ldots, V_k$ be subspaces of a vector space V. Consider $W = V_1 \cap V_2 \cap \cdots \cap V_k$.

(a) Because V_1 is a subspace, we know $\mathbf{0} \in V_1$. Similarly, $\mathbf{0} \in V_i$ for each $i = 2, \ldots k$. Therefore $\mathbf{0} \in V_1 \cap \cdots \cap V_n = W$.

(b) Homework. □