PARTIAL ANSWER SHEET

For more detailed solutions, see similar problems from HW

1. a. T  b. F  c. F  d. F  e. T  f. F

2. a. HW Due 3/28

b. Suppose $S: U \to V$, $T: V \to W$ linear. Then $\operatorname{Ker} S \subseteq \operatorname{Ker}(T \circ S)$

Proof

Let $\vec{u} \in \operatorname{Ker} S$, i.e. $S(\vec{u}) = \vec{0}$.

Then $(T \circ S)(\vec{u}) = T(S(\vec{u})) = T(\vec{0}) = \vec{0} \Rightarrow \vec{u} \in \operatorname{Ker} T \circ S$

$\therefore \operatorname{Ker} S \subseteq \operatorname{Ker} T \circ S$ $\square$

3. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\simeq} \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$

Ker T basis $\{(-1/2, -1/2, 1)\}$

$\operatorname{Im} T = \mathbb{R}^2$ Basis $\{(3,1), (1,-1)\}$ (or $\{(1,0), (0,1)\}$)

4. Checking linear - Many HW problems

If $B = \{1, x, x^2\}$, $C = \{1, x, x^2, x^3\}$  $S: \mathbb{P}_2 \to \mathbb{P}_3$, $S(p) = p - 2xp$

$$S_{CB} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\simeq} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\operatorname{Ker} S_{CB} = \vec{0} \Longrightarrow \operatorname{Ker} S = \vec{0}$

$\operatorname{Im} S_{CB} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\} \Longrightarrow \operatorname{Im} S = \operatorname{span}\{1-2x, x-2x^2, x^2-2x^3\}$

5. Let $D = \{1, 1+x, 1+x+x^2\}$

a. $$T_{DD} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\simeq} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

b. $\operatorname{Ker} T_{DD} = \vec{0} \Rightarrow \operatorname{Ker} T = \vec{0}$

$\operatorname{Im} T_{DD} = \mathbb{R}^3 \Rightarrow \operatorname{Im} T = \mathbb{P}_2$

Let $B = \{1, x, x^2\}$  $P_{BD} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

c. $$T_{BB} = P_{BD} T_{DD} P_{DB} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Note: I have B same basis as prev problem

d. $(S \circ T)_{CB} = S_{CB} T_{BB} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

e. T is ISO since $\text{Ker } T = \vec{0}$, $\text{Im } T = P_2$.

$(T_{DD})^{-1}$ $\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$ $T_{DD}^{-1}$

$(T^{-1})_{DD} = T_{DD}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

ie $T(1) = 1 + x + x^2$
$T(1+x) = 1$
$T(1+x+x^2) = 1 + x$

or could say $T_{BB} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

$T_{BB}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

7. $B = \{(1,2,3), (2,-1,0), (3,1,-1)\}$  $[(9,8,3)]_B = ?$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \simeq \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$  $[9,8,3]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

8. Suppose $T: V \to W$ linear, $\dim V = 5$, $\dim \text{Ker } T = 2$.
Then $\exists\ U \leq V$ subspace with $\dim U = 3$, and $\dim T(U) = 3$.

Proof Outline

Let $\{v_1, v_2\}$ be basis for $\text{Ker } T$, and $\{v_1, v_2, v_3, v_4, v_5\}$ basis for $V$. (using Expansion Thm).

Define $U = \text{span}\{v_3, v_4, v_5\}$.
(Why is $\dim U = 3$?)
Check $\dim T(U) = 3$.