TEST 2 REVIEW

MATH 309, SECTION 6

You should remember the definitions and have a working knowledge of the following concepts already covered: subspaces, linear independence, span, basis, how to solve linear systems, parameterize solution spaces, find a basis for vector spaces/subspaces.

You need to explicitly know: Linear maps, image, kernel, one-to-one, onto, isomorphism, inverses, composition of maps, coordinates, matrix of a linear function with respect to a basis, change of basis.

(1) True or False:
   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V, and the linear map $T : V \to W$ is an isomorphism, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ then is a basis of W.
   (b) If $T : V \to W$ is linear, then $\operatorname{Ker} T$ is a subspace of W.
   (c) If T is not linear, then T is onto.
   (d) Let A be a square matrix. If $A^2 = \mathbf{0}$, then $A = \mathbf{0}$.
   (e) If A is invertible, and $AB = \mathbf{0}$, then $B = \mathbf{0}$.
   (f) Suppose $A \in \mathbb{M}(m, n), B \in \mathbb{M}(n, m)$, and $AB = I_m$. Then $BA = I_n$.

(2) Perform the following proofs:
   (a) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent in V, and $T : V \to W$ is a one-to-one linear map, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a linearly independent subset of W.
   (b) Suppose $S : U \to V$ and $T : V \to W$ are linear maps which are both onto. Prove that $T \circ S$ is onto.
   (c) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. Show that $\operatorname{Ker} S \subset \operatorname{Ker}(T \circ S)$.
   (d) Suppose $S : U \to V$ and $T : V \to W$ are linear maps. If $\operatorname{Image}(S) \subset \operatorname{Ker} T$, then $T \circ S = \mathbf{0}$.

(3) Find a basis for the kernel and image of T, where $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear map whose matrix (relative to the standard basis) is

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}.$$

(4) Let $S : \mathbb{P}_2 \to \mathbb{P}_3$ be the map given by $S(p) = p - 2xp$. Show S is linear. Write the matrix of S relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$. Find the kernel and the image of S.

(5) Suppose $T : \mathbb{P}_2 \to \mathbb{P}_2$ is linear and satisfies

$$T(1) = 1 + x, \quad T(1 + x) = 1 + x + x^2, \quad T(1 + x + x^2) = 1.$$

   (a) Write the matrix of T relative to the basis $\{1, 1 + x, 1 + x + x^2\}$.
   (b) Calculate the kernel and image of T.

(c) Find the matrix of T relative to the basis $\{1, x, x^2\}$. (You may leave your answer as a product of matrices and inverses.)
(d) Where S is the linear map from the previous problem, calculate the matrix of $S \circ T$ relative to the standard bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$. (You may leave your answer as a product of matrices and inverses.)
(e) Is T an isomorphism? If so, find T^{-1}.

(6) Show the following are linear maps: (*Insert your own favorite linear map*).
(7) Let $B = \{(1, 2, 3), (2, -1, 0), (3, 1, -1)\}$ be a basis for $\mathbb{R}^3$. Find the B-coordinates of $(9, 8, 3)$; i.e. find $[(9, 8, 3)]_B$.
(8) Suppose that $T : V \to W$ is a linear map, $\dim V = 5$, and $\dim \operatorname{Ker} T = 2$. Show there exists a 3-dimensional subspace $U \subseteq V$ such that $T(U)$ is a 3-dimensional subspace of W. (Hint: start with a basis for $\operatorname{Ker} T$ and form a basis for V.)