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The Art of Proof

Basic Training for Deeper Mathematics

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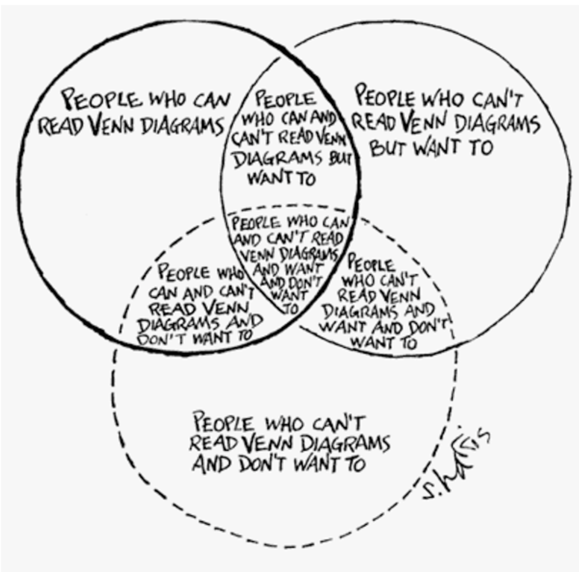
Chapter 5
Underlying Notions in Set Theory



Before You Get Started. Come up with real-life examples of sets that behave like the ones in the above picture; for example, you might think of friends of yours that could be grouped according to certain characteristics—those younger than 20, those who are female, etc. Carefully label your picture. Make your example rich enough that all of the regions in the picture have members.

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5.1 Subsets and Set Equality

Pay attention to the difference between an element *of a set S and a* subset *of a set S. A subset of S is a set, while an element of S is one of the things that is in the set S.*

A **set** is a collection of "things" usually called **elements** or **members**. The notation $x \in A$ means that x is a member of (an element of) the set A. The negation of $x \in A$ is written $x \notin A$. It means that x is not a member of A.

As we saw in Chapter 2, we write $A \subseteq B$ (A is a subset of B) when every member of A is a member of B, i.e.,

$$x \in A \;\Rightarrow\; x \in B\,.$$

The symbol $\supseteq$ is also used when we want to read from right to left: $B \supseteq A$ means $A \subseteq B$.

Proposition 5.1. *Let A, B, C be sets.*

(i) $A \subseteq A$.

(ii) *If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

Proof. (i) $A \subseteq A$ means "if $x \in A$ then $x \in A$," which is a true statement.

(ii) Assume $A \subseteq B$ and $B \subseteq C$. We need to show that if $x \in A$ then $x \in C$. Given $x \in A$, $A \subseteq B$ implies that $x \in B$. Since $B \subseteq C$, this implies that $x \in C$. □

Another concept, already introduced in Chapter 2, is set equality: We write $A = B$ when A and B are the same set, i.e., when A and B have precisely the same members, i.e., when

$$A \subseteq B \qquad \text{and} \qquad B \subseteq A\,. \tag{5.1}$$

Note that equality of sets has a different flavor from equality of numbers. To prove that two sets are equal often involves hard work—we have to establish the two subset relations in (5.1).

Sometimes the same set can be described in two apparently different ways. For example, let A be the set of all integers of the form $7m+1$, where $m \in \mathbf{Z}$, and let B be the set of all integers of the form $7n-6$, where $n \in \mathbf{Z}$. We write this as

$$A = \{7m+1 : m \in \mathbf{Z}\} \qquad \text{and} \qquad B = \{7n-6 : n \in \mathbf{Z}\}\,.$$

This proposition might be too simple to be interesting. We have included it to illustrate how one proves that two sets are equal.

Proposition 5.2. $\{7m+1 : m \in \mathbf{Z}\} = \{7n-6 : n \in \mathbf{Z}\}\,.$

Proof. We must prove that $A \subseteq B$ and $B \subseteq A$.

The first statement means $x \in A \Rightarrow x \in B$. So let $x \in A$. Then, for some $m \in \mathbf{Z}$, $x = 7m+1$. But $7m+1 = 7(m+1)-6$, and so we can set $n = m+1$, which gives $x = 7n-6$; thus $x \in B$. This proves $A \subseteq B$.

Conversely, let $x \in B$. Then, for some $n \in \mathbf{Z}$, $x = 7n - 6$. But $7n - 6 = 7(n-1) + 1$; setting $m = n - 1$ gives $x = 7m + 1$, and so $x \in A$. This proves $B \subseteq A$ and establishes our desired set equality. □

Template for proving $A = B$. Prove that $A \subseteq B$ and $B \subseteq A$.

Project 5.3. Define the following sets:

$$\begin{aligned}
A &:= \{3x : x \in \mathbf{N}\}, \\
B &:= \{3x + 21 : x \in \mathbf{N}\}, \\
C &:= \{x + 7 : x \in \mathbf{N}\}, \\
D &:= \{3x : x \in \mathbf{N} \text{ and } x > 7\}, \\
E &:= \{x : x \in \mathbf{N}\}, \\
F &:= \{3x - 21 : x \in \mathbf{N}\}, \\
G &:= \{x : x \in \mathbf{N} \text{ and } x > 7\}.
\end{aligned}$$

Determine which of the following set equalities are true. If a statement is true, prove it. If it is false, explain why this set equality does not hold.

(i) $D = E$.

(ii) $C = G$.

(iii) $D = B$.

The sets A and F will make an appearance in Project 5.11.

Here are some facts about equality of sets:

Proposition 5.4. *Let A, B, C be sets.*

(i) $A = A$.

(ii) *If $A = B$ then $B = A$.*

(iii) *If $A = B$ and $B = C$ then $A = C$.*

These three properties should look familiar—we mentioned them already in Section 1.1 when we talked about equality of two integers. We called the properties *reflexivity*, *symmetry*, and *transitivity*, respectively.

We will see these properties again in Section 6.1.

Project 5.5. When reading or writing a set definition, pay attention to what is a variable inside the set definition and what is not a variable. As examples, how do the following pairs of sets differ?

(i) $S := \{m : m \in \mathbf{N}\}$ and $T_m := \{m\}$ for a specified $m \in \mathbf{N}$.

(ii) $U := \{my : y \in \mathbf{Z}, m \in \mathbf{N}, my > 0\}$ and $V_m := \{my : y \in \mathbf{Z}, my > 0\}$ for a specified $m \in \mathbf{N}$.

The subscripts on T_m, V_m, W_m are not necessary, but this notation is often useful to emphasize the fact that m is a constant.

(iii) V_m and $W_m := \{my : y \in \mathbf{Z}, y > 0\}$ for a specified $m \in \mathbf{Z}$.

Find the simplest possible way of writing each of these sets.

The **empty set**, denoted by $\varnothing$, has the feature that $x \in \varnothing$ is never true. We allow ourselves to say *the* empty set because there is only one set with this property:

Proposition 5.6 asserts the uniqueness of $\varnothing$. The existence of $\varnothing$ is one of the hidden assumptions mentioned in Section 1.4.

Proposition 5.6. *If the sets $\varnothing_1$ and $\varnothing_2$ have the property that $x \in \varnothing_1$ is never true and $x \in \varnothing_2$ is never true, then $\varnothing_1 = \varnothing_2$.*

Proof. Assume that the sets $\varnothing_1$ and $\varnothing_2$ have the property that $x \in \varnothing_1$ is never true and $x \in \varnothing_2$ is never true. Suppose (by means of contradiction) that $\varnothing_1 \neq \varnothing_2$, that is, either $\varnothing_1 \not\subseteq \varnothing_2$ or $\varnothing_1 \not\supseteq \varnothing_2$. We first consider the case $\varnothing_1 \not\subseteq \varnothing_2$. This means there is some $x \in \varnothing_1$ such that $x \notin \varnothing_2$. But that cannot be, since there is no $x \in \varnothing_1$. The other case, $\varnothing_1 \not\supseteq \varnothing_2$, is dealt with similarly. □

Proposition 5.7. *The empty set is a subset of every set, that is, for every set S, $\varnothing \subseteq S$.*

Project 5.8. Read through the proof of Proposition 5.1 having in mind that A is empty. Then there exists no x that is in A. Do you see why the proof still holds?

5.2 Intersections and Unions

The **intersection** of two sets A and B is

$$A \cap B = \{x : x \in A \text{ and } x \in B\} .$$

The **union** of A and B is

$$A \cup B = \{x : x \in A \text{ or } x \in B\} .$$

The set operations $\cap$ and $\cup$ give us alternative ways of writing certain sets. Here are two examples:

When two sets A and B satisfy $A \cap B = \varnothing$, we say that A and B are disjoint.

Example 5.9. $\{3x + 1 : x \in \mathbf{Z}\} \cap \{3x + 2 : x \in \mathbf{Z}\} = \varnothing$.

Example 5.10.
$\{2x : x \in \mathbf{Z}, 3 \leq x\} = \{x \in \mathbf{Z} : 5 \leq x\} \cap \{x \in \mathbf{Z} : x \text{ is even}\}$.

Project 5.11. This is a continuation of Project 5.3, and so the following names refer to the sets defined in Project 5.3. Again, determine which of the following set equalities are true. If a statement is true, prove it. If it is false, explain why this set equality does not hold.

(i) $A \cap E = B$.

(ii) $A \cap C = B$.

(iii) $E \cap F = A$.

Project 5.12. Determine which of the following statements are true for all sets A, B, and C. If a double implication fails, determine whether one or the other of the possible implications holds. If a statement is true, prove it. If it is false, provide a counterexample.

"Providing a counterexample" here means coming up with a specific example of a set triple A,B,C that violates the statement.

(i) $C \subseteq A$ and $C \subseteq B \iff C \subseteq (A \cup B)$.

(ii) $C \subseteq A$ or $C \subseteq B \iff C \subseteq (A \cup B)$.

(iii) $C \subseteq A$ and $C \subseteq B \iff C \subseteq (A \cap B)$.

(iv) $C \subseteq A$ or $C \subseteq B \iff C \subseteq (A \cap B)$.

For two sets A and B, we define the **set difference**

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Another commonly used notation for set difference is $A \setminus B$.

Given a set $A \subseteq X$, we define the **complement** of A in X to be $X - A$. If the bigger set X is clear from the context, one often writes A^c for the complement of A (in X).

Example 5.13. Recall that the even integers are those integers that are divisible by 2. The **odd** integers are defined to be those integers that are not even. Thus the set of odd integers is the complement of the set of even integers.

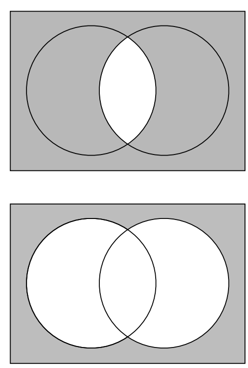
Proposition 5.14. *Let $A, B \subseteq X$.*

$$A \subseteq B \qquad \textit{if and only if} \qquad B^c \subseteq A^c.$$

Theorem 5.15 (De Morgan's laws). *Given two subsets $A, B \subseteq X$,*

$$(A \cap B)^c = A^c \cup B^c \qquad \textit{and} \qquad (A \cup B)^c = A^c \cap B^c.$$

In words: the complement of the intersection is the union of the complements and the complement of the union is the intersection of the complements.



Here are two pictures of De Morgan's equalities.

Project 5.16. Someone tells you that the following equalities are true for all sets A, B, C. In each case, either prove the claim or provide a counterexample.

(i) $A - (B \cup C) = (A - B) \cup (A - C)$.

(ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

As another example of a recursive construction, we invite you to explore unions and intersections of an arbitrary number of sets.

Project 5.17 (Unions and intersections). Given sets $A_1, A_2, A_3, \ldots$, develop recursive definitions for

$$\bigcup_{j=1}^{k} A_j \qquad \text{and} \qquad \bigcap_{j=1}^{k} A_j \, .$$

Find and prove an extension of De Morgan's laws (Theorem 5.15) for these unions and intersections.

Logical paradoxes arise when one treats the "set of all sets" as a set. The "set" R in Project 5.18 is an indication of the problem. These logical issues do not cause difficulties in the mathematics discussed in this book.

Proposition 5.7 says that the empty set $\varnothing$ is "extreme" in that it is the smallest possible set. Thus $S \neq \varnothing$ if and only if there exists an x such that $x \in S$. One would like to go to the other extreme and define a set that contains "everything"; however, there is no such set.

Project 5.18. Let $R = \{X : X \text{ is a set and } X \notin X\}$. Is the statement $R \in R$ true or false?

5.3 Cartesian Products

Let A and B be sets. From them we obtain a new set

$$A \times B := \{(a,b) : a \in A \text{ and } b \in B\} \, .$$

Cartesian products are named after René Descartes (1596–1650), who used this concept in his development of analytic geometry.

We call (a,b) an **ordered pair**. The set $A \times B$ is called the **(Cartesian) product** of A and B. It is the set of all ordered pairs whose first entry is a member of A and whose second entry is a member of B.

Example 5.19. $(3,-2)$ is an ordered pair of integers, and $\mathbf{Z} \times \mathbf{Z}$ denotes the set of all ordered pairs of integers. (Draw a picture.)

Notice that when $A \neq B$, $A \times B$ and $B \times A$ are different sets.

Proposition 5.20. *Let A, B, C be sets.*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Project 5.21. Let A, B, C, D be sets. Decide whether each of the following statements is true or false; in each case prove the statement or give a counterexample.

(i) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

(ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

5.4 Functions

We come to one of the most important ideas in mathematics. There is an informal definition and a more abstract definition of the concept of a function. We give both.

First Definition. A **function** consists of

- a set A called the **domain** of the function;
- a set B called the **codomain** of the function;
- a rule f that assigns to each $a \in A$ an element $f(a) \in B$.

A useful shorthand for this is $f : A \to B$.

This notation suggests that the function f picks up each $a \in A$ and carries it over to B, placing it precisely on top of an element $f(a) \in B$.

Example 5.22. $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(n) = n^3 + 1$.

Example 5.23. Every sequence $(x_j)_{j=1}^{\infty}$ is a function with domain $\mathbf{N}$, where we write x_j instead of $f(j)$.

The **graph** of $f : A \to B$ is

$$\Gamma(f) = \{(a,b) \in A \times B : b = f(a)\}.$$

Sometimes mathematicians ask whether a function is well defined. *What they mean is this: "Does the rule you propose really assign to each element of the domain one and only one value in the codomain?"*

Project 5.24. Discuss how much of this concept coincides with the notion of the graph of $f(x)$ in your calculus courses.

A possible objection to our first definition is that we used the undefined words *rule* and *assigns*. To avoid this, we offer the following alternative definition of a function through its graph:

Second Definition. A **function** with **domain** A and **codomain** B is a subset Γ of $A \times B$ such that for each $a \in A$ there is one and only one element of Γ whose first entry is a. If $(a,b) \in \Gamma$, we write $b = f(a)$.

Project 5.25. Discuss our two definitions of function. What are the advantages and disadvantages of each? Compare them with the definition you learned in calculus.

Example 5.26. A binary operation on a set A is a function $f : A \times A \to A$. For example, Axiom 1.1 could be restated as follows: There are two functions plus: $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ and times: $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ such that for all integers m, n, and p,

$$\begin{aligned} \text{plus}(m,n) &= \text{plus}(n,m) \\ \text{plus}(\text{plus}(m,n),p) &= \text{plus}(m,\text{plus}(n,p)) \\ \text{times}(m,\text{plus}(n,p)) &= \text{plus}(\text{times}(m,n),\text{times}(m,p)) \\ \text{times}(m,n) &= \text{times}(n,m) \\ \text{times}(\text{times}(m,n),p) &= \text{times}(m,\text{times}(n,p))\,. \end{aligned}$$

Review Question. Do you understand the difference between $\in$ and $\subseteq$?

Weekly reminder: Reading mathematics is not like reading novels or history. You need to think slowly about every sentence. Usually, you will need to reread the same material later, often more than one rereading.

This is a short book. Its core material occupies about 140 pages. Yet it takes a semester for most students to master this material. In summary: read line by line, not page by page.