Tutorial Worksheet, 01/13/2016

Instructions: Please work in groups of 3 or 4 students. Please work with students who will attend the same recitation section. You do turn this worksheet in at the end of class; instead, attendance will be recorded so that you get credit for participating in this activity.

Solutions to 01/11: An Introduction to Mathematical Modeling

1. Suppose that the density of a bacteria population is proportional to B_t, where t is a *time index* indicating 16 minute time intervals (i.e. $t = 0, 1, 2$ corresponds to the initial measurement, a measurement after 16 minutes, a measurement after 32 minutes).

   As explained on p.3 of our textbook, B_t is, more precisely, the *absorbance* of light passing through the bacteria colony and measured by a *spectrophotometer*.

   How might you measure the population of a bacteria colony?

   There are many possible answers. The following Wikipedia article has an extensive discussion of cell counting:

   `https://en.wikipedia.org/wiki/Cell_counting`

   Follow-up Questions:

   (a) What is absorbance?
   (b) How is a spectrophotometer used to measure the density of a bacteria population?

2. Below is the completed table.

Table 1: Measurements of bacterial density at pH 6.25.

Time (min)	Time Index t	Population Density B_t	Population Change / Unit Time $(B_{t+1} - B_t)/1$
0	0	0.022	0.014
16	1	0.036	0.024
32	2	0.060	0.041
48	3	0.101	0.068
64	4	0.169	0.097
80	5	0.266	undefined

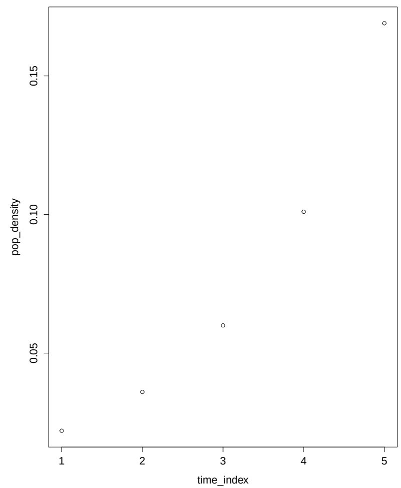
   Follow-up Questions:

   (a) What time (in minutes) corresponds to a time index of $t = 10$?

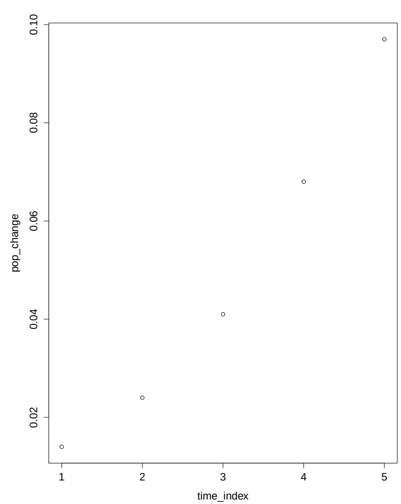
(b) What equation expresses the relationship between the time, T, and the time index, t?

3. Below are three graphs related to Table 1.

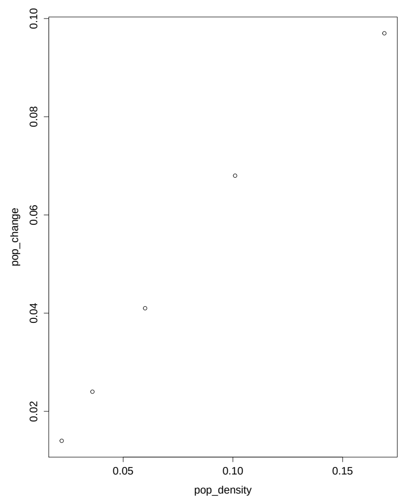
(a) Population Density vs. Time Index



(b) Population Change per Unit Time vs. Time Index



(c) Population Change per Unit Time vs. Population Density



Follow-up Question: Which of the following adjectives is descriptive of each of the graphs above?

(a) linear

(b) quadratic

(c) exponential

Solutions to 01/12: Graphing and Plotting

1. The graphs of the equations below are described.

   (a) $Y = 4 - \frac{3}{2}X$. The graph is a line with Y-intercept 4 and slope $-3/2$.

   (b) $Q = P^2$. The graph is a parabola which opens upward, passes through $(0, 0)$, and is symmetric about the Q-axis.

   (c) $x^2 + y^2 = 4$. The graph is a circle of radius 2 centered at the origin, $(0, 0)$.

   (d) $P = 2^t$. The graph is the graph of the exponential function $P = f(t) = 2^t$. The values of P are always positive. The graph is increasing.

   (e) $A = (1/2)^t$. The graph is the graph of the exponential function $A = g(t) = 2^{-t}$. The values of A are always positive. The graph is decreasing.

Follow-up Questions:

(a) How can one use the value of the slope of a line to determine whether the line is increasing or decreasing?

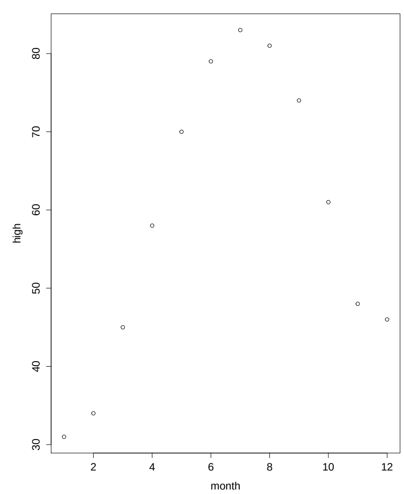
(b) What is the line of symmetry of the graph of the equation $y = (x - 2)^2$?

(c) How can you tell from an equation whether or not a parabola opens upward or opens downward?

2. Below is a plot of the data in the table below.

Table 2: Average daily high temperature in Detroit

Month (1 = Jan.)	1	2	3	4	5	6	7	8	9	10	11	12
High (°F)	31	34	45	58	70	79	83	81	74	61	48	46



3. We will try to find values of A, B, C, and D so that the graph of the equation $y = A\cos(B(x - C) + D$ fits the data in Table 2.

(a) Warm-up exercises: the graphs of each of the following equations is described below.

i. $y = \cos x$. The graph of the cosine wave is familiar from your previous course work. We will discuss sine and cosine waves in more detail below. It has amplitude $A = 1$ and period $\mathrm{T} = 2\pi$.

ii. $y = 2\cos x$. This is a cosine wave with amplitude $A = 2$ and period $\mathrm{T} = 2$.

iii. $y = \cos(2x)$. This is a cosine wave with amplitude $A = 1$ and period $\mathrm{T} = \pi$.

iv. $y = \cos(x - \frac{\pi}{2})$. This is a cosine wave with amplitude $A = 1$ and period $\mathrm{T} = 2\pi$, and it has been shifted by $\pi/2$ to the right. When you graph this, you will see that it coincides with the graph of $y = \sin x$. Indeed, it is true that $\cos(x - \frac{\pi}{2}) = \sin x$ for every real number x.

v. $y = \cos x + 2$. This is a cosine wave with amplitude $A = 1$ and period $\mathrm{T} = 2\pi$, and it has been shifted vertically by 2.

(b) Discuss how you might determine values of A, B, C, and D which seem to fit the data in Table 2.

Follow-up Question: What is your strategy for determining the values of A, B, C, and D?

01/13: Sine and Cosine Waves

A **sine wave** is a graph of an equation of the form

$$y = A\sin(\omega(x - \varphi)).$$

The following quantities are characteristics of this sine wave:

1. A is the **amplitude**.
2. ω is the **angular frequency**
3. The **period** is $\mathrm{T} = \frac{2\pi}{\omega}$.
4. φ is the **horizontal shift** or **phase shift**

Caution: Physicists and engineers use a different convention: the quantity $\omega\phi$ is called the "phase shift."

A **cosine wave** is defined analogously: just replace sin with cos in the definition above..

The following equations,

$$\cos(x - \frac{\pi}{2}) = \sin x \tag{1}$$

$$\sin(x + \frac{\pi}{2}) = \cos x \tag{2}$$

imply that every sine wave is a cosine wave and vice versa. A vertical translation of a sine/cosine wave is also called a sine/cosine wave.

Review of Transformations:

1. The graph of $y = f(x-h)$ is the same as the graph of $y = f(x)$ translated horizontally to the right by h.
2. The graph of $y = f(x)+v$ is the same as the graph of $y = f(x)$ translated vertically upward by v.

3. The graph of $y = sf(x)$ is the same as the graph of $y = f(x)$ stretched (relative to the x-axis) vertically by a factor of s if $s > 0$.

4. The graph of $y = f(cx)$ is the same as the graph of $y = f(x)$ compressed (relative to the y-axis) horizontally by a factor of c if $c > 0$.

Notes:

- If $0 < s < 1$, the vertical stretch is more accurately described as a "vertical compression."
- If $0 < c < 1$, the horizontal compression is more accurately described as a "horizontal stretch."
- If $s < 0$, then the graph is reflected in the y-axis and stretched vertically by a factor of $|s|$.
- If $c < 0$, then the graph is reflected in the x-axis and compressed horizontally by a factor of $|c|$.

Exercise: For each equation below, identify the amplitude, angular frequency, period, and phase shift. Then graph each equation.

1. $y = \sin(2x + \pi)$

2. $y = 3\cos\left(\frac{x}{2}\right)$

3. $y = 10\sin\left(\frac{\pi}{10}x\right) + 3$

4. $y = \frac{1}{2}\cos\left(4x + \pi\right) - 1$

Exercise: Suppose that a buoy is bobbing up and down in a harbor. The buoy reaches a maximum elevation of 0.5 meters above sea level and a minimum elevation of 0.5 meters below sea level. The time it takes for the buoy to descend from its maximum elevation to its minimum elevation is 0.75 seconds. Write an equation which models the elevation of the buoy as a function of time.

Exercise: Find values of A, B, C, and D so that the graph of

$$y = A\cos\left(B(x - C) + D\right.$$

fits the data in Table 2.