Tutorial Worksheet, 01/20/2016

Instructions: Please work in groups of 3 or 4 students. Please work with students who will attend the same recitation section. You do turn this worksheet in at the end of class; instead, attendance will be recorded so that you get credit for participating in this activity.

Solutions to 01/13 and 01/19 Worksheets:

Order of Transformations Exercises:

1. Let $f(x) = \sin x$. Sketch the graph of $y = f(x)$. Make a careful sketch. You should clearly indicate the zeroes, i.e. the points on the graph where it touches the x-axis. You will use this same function in problems 2–4 below.

   Solution: The zeroes occur precisely when $x \in \{0, \pm\pi, \pm 2\pi, \dots\}$. Another way to write this is that the zeroes occur precisely at x-values of the form $x = n\pi$, where n is an integer. The sine wave has period $\mathrm{T} = 2\pi$, angular frequency $\omega = 1$, and amplitude $A = 1$. It has a maximum value at $x = \frac{\pi}{2}$. It has a minimum value at $x = \frac{3\pi}{2}$. These maximum and minimum values occur once in each period. Thus, there is a maximum value precisely when x has the form $x = \frac{\pi}{2} + 2n\pi$, where n is an integer.

2. Sketch a graph of $y = f(x)$ that has, first, been translated to the right by $\pi/2$ and then, second, has its period compressed to π.

   Solution: The translation shifts the zeroes to when $x = \frac{\pi}{2} + n\pi$, where n is an integer. The compression scales each of these by a factor of two, i.e. each is scaled by $1/2$. Thus, the zeroes occur when $x = \frac{\pi}{4} + \frac{n\pi}{2}$, where n is an integer. Similarly, the maximum values are shifted to when $x = \pi + 2n\pi$, where n is an integer. Then these are then compressed so that the maxima occur when $x = \frac{\pi}{2} + n\pi$, where n is an integer.

3. Sketch a graph of $y = f(x)$ that has, first, had its period compressed to π and then, second, has been translated to the right by $\pi/2$.

   Solution: The compression moves the zeros to when $x = \frac{n\pi}{2}$, where n is an integer. The translation moves these to $x = \frac{\pi}{2} + \frac{n\pi}{2}$, where n is an integer. This can be re-expressed as $x = \frac{n\pi}{2}$, where n is an integer. The compression moves the maxima to when $x = \frac{\pi}{4} + n\pi$, where n is an integer. The translation moves these to $x = \frac{3\pi}{4} + n\pi$, where n is an integer.

4. Consider the following two equations below. Which one did you graph in problem 2? Which one did you graph in problem 3?

   (a) $y = \sin\left(2(x - \frac{\pi}{2})\right)$

   (b) $y = \sin\left(2x - \frac{\pi}{2}\right)$

Solution: The zeroes of the first equation occur when

$$2(x - \frac{\pi}{2}) = n\pi,$$

where n is an integer. Solving, we see that the zeroes occur when $x = \frac{\pi}{2} + \frac{n\pi}{2}$. From this we can deduce that the equation $y = \sin(2(x - \frac{\pi}{2}))$ describes the sketch in problem 3 above: it is a sine wave that has first been compressed by a factor of two and then translated to the right by $\pi/2$.

Follow-up Problem: Try to do a similar analysis of the second equation above to determine its zeroes using algebra. Then compare your answer with the solution to problem 2 above.

Sine & Cosine Wave Exercises: For each equation below, identify the amplitude, angular frequency, period, and phase shift. Then graph each equation. Use the technique above: write the wave in the form $y = \sin(\omega(x - \varphi))$ and, first, graph $y = \sin(\omega x)$ and then, second, translate this graph to the right by φ. (Use the analogous method in the case of a cosine wave.)

1. $y = \sin(2x + \pi)$
2. $y = 3\cos(\frac{x}{2})$
3. $y = 10\sin(\frac{\pi}{10}x) + 3$
4. $y = \frac{1}{2}\cos(4x + \pi) - 1$

Solutions: We will go over the solutions together now. I will post solutions on our course web page.

A Bobbing Buoy: Suppose that a buoy is bobbing up and down in a harbor. The buoy reaches a maximum elevation of 0.5 meters above sea level and a minimum elevation of 0.5 meters below sea level. The time it takes for the buoy to descend from its maximum elevation to its minimum elevation is 0.75 seconds. Write an equation which models the elevation of the buoy as a function of time.

Solution: Let E be the elevation of the buoy at time t. Here, t is the number of seconds after a measurement where the buoy is at its maximum elevation. With this choice of initial time, we have that $E = A\cos\omega t$. The amplitude, A, is equal to 0.5. The angular frequency, ω, can be determined from the formula $\omega = \frac{2\pi}{\mathrm{T}}$. We are given that $\frac{1}{2}\mathrm{T} = 0.75$ so that $\mathrm{T} = 1.5 = \frac{3}{2}$ and so $\omega = \frac{4\pi}{3}$. Therefore,

$$E = 0.5\cos(4\pi t/3)$$

models the elevation of the buoy as a function of time.

Modeling Temperature Exercise: Find values of A, B, C, and D so that the graph of

$$y = A\cos(B(x - C) + D$$

Table 1: Average daily high temperature in Detroit

Month (1 = Jan.)	1	2	3	4	5	6	7	8	9	10	11	12
High (°F)	31	34	45	58	70	79	83	81	74	61	48	46

fits the data in Table 1.

Solution: We can approximate the amplitude as one half of the distance between the maximum and minimum temperatures. So, we let

$$A = (1/2)(83 - 31) = 26.$$

We need to shift our wave vertically so that the temperatures oscillate between 83 and 31. So, we need to shift vertically by the average of these two values. Thus,

$$D = (1/2)(83 + 31) = 57.$$

If we let $x = 1$ correspond to January and $x = 2$ to February, etc., then the lowest temperature occurs at time $x = 1$ and the period of our wave is 12 (since $x = 13$ means January of the next year). From this we have that the period is T = 12 and so the angular frequency is $\omega = 2\pi/\mathrm{T} = \pi/6$.

Finally, to determine the phase shift, we first choose to use a cosine wave. We want the maximum value to occur at $x = 7$. Therefore, we let $\varphi = 7$. Thus,

$$y = 26\cos\left(\frac{\pi}{6}(x - 7)\right) + 57$$

models the temperature. We can check that the phase shift is correct by observing that, at $x = 7$, the value of y is $26 + 57 = 83$ and that, at $x = 1$, the value of y is $26\cos(\pi) + 57 = -26 + 57 = 31$.

01/20: A Mathematical Model of Sunlight Depletion

The following "explore" exercises are from our textbook. Please work on these with the other members of your group.

Explore 1.3.1

a Suppose the light intensity at the surface is I_0 and that at a depth of 10 meters it is half of this. What do you expect the light intensity to be at a depth of 20 meters?

b Sketch a proposed graph of light intensity versus depth.

Explore 1.3.2

Assume that at the surface I_0 is 400 watts per square meter. And assume that 10% of the light is absorbed in each layer of water that is 2 meters thick, i.e. 2 meters deep. What is the intensity of light at depths of $2, 4, 6, \ldots, 20$ meters? Plot this data and compare this graph to the one you drew in part (b) of Explore 1.3.1 above.

Exercise: Use a complete sentence to describe the meaning of the following dynamic equation

$$I_d - I_{d+1} = f\, I_d,$$

where I_d is the intensity of light at depth d and f is the fraction of light absorbed in a layer of water of some fixed thickness.

Exercise: Solve the dynamic equation in the previous exercise.