Tutorial Worksheet, 02/24/2016

Instructions: Please work in groups of 3 or 4 students. Please work with students who will attend the same recitation section. You do not turn this worksheet in at the end of class; instead, attendance will be recorded so that you get credit for participating in this activity.

02/24: Properties of Limits and Derivatives

Linearity. The operation of taking the derivative of a function is a *linear operation*. This means the following

- $(F(x) + G(x))' = F'(x) + G'(x)$ and
- $(c \cdot F(x))' = c \cdot F'(x)$ for any constant c.

Power Rule. We have seen that for any positive integer n, if $F(x) = x^n$, then $F'(x) = nx^{n-1}$. (In fact, this rule is true for any number n, not just for integers; we'll discuss this fact later.)

Exercises. Use the power rule and the fact that the derivative is a linear operator to compute the derivative of each function below.

1. $F(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

2. $G(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$

3. Expand the following and then use the properties above to compute the derivative of $H(x)$:
$$H(x) = (1 + x + x^2)^2.$$

4. Recompute the derivative of $H(x) = (1+x+x^2)^2$ by using the chain rule. The chain rule states that

$$(F(G(x)))' = F'(G(x)) \cdot G'(x).$$

The following steps will guide you through the computation.

   (a) Express $H(x)$ as the composite of two functions $F(x)$ and $G(x)$ so that $H(x) = F(G(x))$.

   (b) Compute $F'(x)$. Compute $F'(G(x))$.

   (c) Compute $G'(x)$.

   (d) Compute $F'(G(x)) \cdot G'(x)$ and expand your answer as much as possible. Compare this answer to the answer you obtained in the previous exercise.

5. (Tricky!) Compute $L'(0)$ where

$$L(x) = (1+x)(1+x+x^2)(1+x+x^2+x^3)\cdots(1+x+x^2+x^3+\cdots x^{100}).$$

Hint: You don't need to determine $L'(x)$ exactly. You only need to figure out what its value at $x = 0$.

Applications. Since the derivative of $F'(a)$ is the rate of change of $F(x)$ with respect to x at $x = a$ and since $F'(a)$ is also the slope of the tangent line to the graph of $F(x)$ at $(a, F(a))$, the derivative can be used to solve some related problems.

Exercises.

1. Suppose that the area (in square millimeters), A, of mold colony is given as a function of the number of days, t, after an initial measurement corresponding to $t = 0$. If $A = 1.25t^2$, at what rate is the area of the colony changing on the third day? Use the correct units for your answer.

2. Determine the equation of the line tangent to the curve $y = 3 - x^2$ at the point on the graph where $x = 1$.

3. Suppose that the height (in meters), h, of a ball is given as a function of time (in seconds), t. The value $t = 0$ represents the time of an initial measurement. When a ball is thrown directly upward, i.e. moves only in a vertical direction, with an initial speed of 20 meters per second and the ball is initially at a height of zero meters (representing ground level), then h has the following description:

$$h(t) = 20t - 4.9t^2.$$

   (a) What are the units of h', the derivative of h with respect to t?

   (b) What is $h(0)$? What is $h'(0)$? Explain why these are reasonable values based on the description above.

   (c) Determine the value of t at which the ball reaches is maximum height.

   (d) Determine the maximum height of the ball.

4. (Challenging!) There are two tangent lines to the graph of $y = x^2$ which pass through the point $(-1, 0)$. Determine the points at which these lines are tangent to the graph.

   Hint: Sketch a picture first. The point $(-1, 0)$ does not lie on the graph of $y = x^2$. Rather, you should draw two lines which pass through $(-1, 0)$ and are tangent to $y = x^2$ at some other points.

5. What shape of curve do you obtain when you unwind thread from a spool? To be more precise, imagine the spool is held in place by a metal rod so that it is free to spin it cannot be moved from side to side. The rod juts out of the chalkboard and the spool is pressed flush against the board. You have tied the loose end of the thread to a piece of chalk. Taking up any slack in the thread, you slowly start to unwind the thread, tracing a path with the chalk as you do so. What do you see? Do you see how this is related to tangent lines?