Tutorial Worksheet, 01/25/2016

Instructions: Please work in groups of 3 or 4 students. Please work with students who will attend the same recitation section. You do turn this worksheet in at the end of class; instead, attendance will be recorded so that you get credit for participating in this activity.

Solutions to 01/20 Worksheet:

01/20: A Mathematical Model of Sunlight Depletion

The following "explore" exercises are from our textbook. Please work on these with the other members of your group.

Explore 1.3.1

a Suppose the light intensity at the surface is I_0 and that at a depth of 10 meters it is half of this. What do you expect the light intensity to be at a depth of 20 meters?

b Sketch a proposed graph of light intensity versus depth.

Solution: The light intensity should be one fourth of I_0 at a depth of 20 meters. A graph of light intensity versus depth will have the shape of a decaying exponential, e.g. the graph of the equation $y = AB^t$, where A and B are constants and $0 < B < 1$.

Explore 1.3.2

Assume that at the surface I_0 is 400 watts per square meter. And assume that 10% of the light is absorbed in each layer of water that is 2 meters thick, i.e. 2 meters deep. What is the intensity of light at depths of $2, 4, 6, \ldots, 20$ meters? Plot this data and compare this graph to the one you drew in part (b) of Explore 1.3.1 above.

Solution: Let I_d be the intensity at depth d. Then

$$I_{d+1} = (0.10)I_d.$$

In other words, at depth 2, the intensity is $0.10 \times 400 = 40$. And at depth 4, the intensity is $0.10 \times 40 = 4$. At depth six the intensity is $0.10 \times 4 = 0.4$, etc. The units are watts per square meter. The intensity of light at a depth of 20 meters is $(0.10)^{10} \times 400 = 4 \times 10^{-8}$. A plot of this data verifies that the intensity decays exponentially as depth increases.

Exercise: Use a complete sentence to describe the meaning of the following dynamic equation

$$I_d - I_{d+1} = f\, I_d,$$

where I_d is the intensity of light at depth d and f is the fraction of light absorbed in a layer of water of some fixed thickness.

Solution: The intensity of light from one layer, I_d, to the next, I_{d+1} *decreases* by a fraction, f, of the intensity of light at the given layer, I_d.

Note: We know that the intensity is decreasing because the change, $I_{d+1} - I_d$ is equal to $-fI_d$, where $0 < f < 1$. So, the change is negative.

Exercise: Solve the dynamic equation in the previous exercise.

Solution: Below are the steps.

1. Given $I_d - I_{d+1} = f\, I_d$.
2. Re-write: $(1-f)I_d = I_{d+1}$
3. The above equation holds for all integers $d > 0$. So, $I_d = (1-f)I_{d-1} = (1-f)^2 I_{d-2}$ and so forth.
4. We conclude that
$$I_d = (1-f)^d I_0.$$

Follow-up Question: Solve Exercise 1.4.7 on p.20. I have added some additional instructions.

Exercise 1.4.7. Light intensities, I_1 and I_2, are measured at depths d in meters in two lakes on two different days and found to be approximately

$$I_1 = 2 \cdot 2^{-0.1d} \quad \text{and} \quad I_2 = 4 \cdot 2^{-0.2d}.$$

a What is the half-life of I_1? In other words, at what depth is the intensity equal to half of its initial intensity.

b What is the half-life of I_2?

c Find a depth at which the two light intensities are the same. How would you solve this problem using algebra?

d Which of the two lakes is the muddiest?

Reminder: Exam I: Our first exam is next Wednesday, 02/03/2016. You should solve all of the exercises in sections 1.1–1.8 to prepare for this exam.

01/25: A Quadratic Model of Mold Growth

Exercise: Solve the following Explore exercises. Please use the handout. (It's a photocopy of pp.22–23 from the text.)

Explore 1.5.1. Using the photographs, measure the areas of the mold for the days 2 and 6 and enter the values into the table in Figure 1.15. The grid lines are at 2 mm intervals. Check your additional data points with points on the graph.

Explore 1.5.2. Find real numbers A and B so that $y = AB^t$ approximates the mold growth data in Figure 1.15. You should choose two data points and insist that these satisfy the equation; then solve for A and B. Similarly, find a parabola $y = at^2 + bt + c$ that approximates the mold data. Draw graphs of the mold data, the exponential function, and the parabola on a single set of axes.