Tutorial Worksheet, 01/26/2016

Instructions: Please work in groups of 3 or 4 students and solve each of the problems below. Your LA can help you. You do not turn these in at the end of class, but your LA will take attendance so that you get credit for participating. We will go over the solutions in the next class. You can spend up to 30 minutes on this worksheet. Afterwards, your LA will give you a quiz. This week's quiz is on properties of exponential and logarithmic functions and half-life and doubling times. Working through the problems below will help to prepare you for the quiz.

01/26: Exponential & Logarithmic Functions

An **exponential function** is a function of the form $y = cb^x$, where b and c are real numbers and $b > 0$. Here y is the dependent variable and x in the independent variable.

The following properties of exponents are useful.

1. $b^x b^y = b^{x+y}$
2. $(b^x)^y = b^{xy}$
3. $b^0 = 1$
4. $b^{-x} = 1/b^x$
5. $b^x/b^y = b^{x-y}$
6. The one-to-one property: $b^x = b^y$ if and only if $x = y$.

Exercises: Use the properties above to write each function in the form $y = cb^x$, where y is the dependent variable and x is the independent variable.

1. $y = (3 \cdot 2^x) \cdot (4 \cdot 2^{x+1})$

2. $y = (3 \cdot 2^x)^3$

3. $y = 2e^x/(4e^{-x})$

Exercise: Solve $P_{t+1} - P_t = rP_t$, where $r > 0$ is a constant, and write the solution equation in the form $P_t = AB^t$.

A **logarithmic function** is a function of the form $y = c\log_b x$, where b and c are real numbers and $b > 0$. As before, y is the dependent variable and x is the independent variable.

The most important property of logarithms is the following one which essentially *defines* the logarithm:

$$y = \log_b x \quad \text{if and only if} \quad x = b^y.$$

Examples.

1. $\log_{10} 1000 = 3$ because $1000 = 10^3$
2. $\log_2 32 = 5$ because $32 = 2^5$
3. $\log_b 1 = 0$ because $1 = b^0$
4. $\log_b b = 1$ because $b = b^1$
5. $\log_b (b^x) = x$ because $b^x = b^x$

The following properties of logarithms are useful.

1. $\log_b (xy) = \log_b x + \log_b y$
2. $\log_b x^y = y \log_b x$
3. $\log_b x^{-1} = -\log_b x$
4. $\log_b (x/y) = \log_b x - \log_b y$

5. $\log_b x = (\log_c x)/(\log_c b)$, for any choice of base $c > 0$

6. $\log_b x = (\ln x)/(\ln b)$

7. The one-to-one property: $\log_b x = \log_b y$ if and only if $x = y$.

Property 5 is perhaps the most mysterious of the formulas above. Here is a derivation of the formula in 6 steps:

1. $y = \log_b x$ if and only if $x = b^y$.

2. $w = \log_c x$ if and only if $x = c^w$.

3. $z = \log_c b$ if and only if $b = c^z$.

4. By (1) and (2), $b^y = c^w$.

5. By (3) and (4), $b^y = (c^z)^y = c^w$. Therefore, $c^{zy} = c^w$.

6. By the one-to-one property for exponential functions, $zy = w$. In other words, $y = w/z$. This proves Property 5.

Exercises: Use the properties of logarithms to write each function in the form $y = \log_b (cx^r)$, where y is the dependent variable and x is the independent variable.

1. $y = 3\log_2 (2x) + 4\log_2 x$

2. $y = 3\log_{10} x - \log_{10} (3x)$

Exercises: Determine the doubling time of each of the exponential functions below. Give both an exact and a numerical answer.

1. $y = 2^{3t}$

2. $y = 10^{0.1t}$

Exercises: Determine the half-life of each of exponential functions below. Give both and exact and a numerical answer.

1. $y = 100 \cdot (0.8)^t$

2. $y = (0.1)^{0.1t}$

Exercise: If $y = cb^x$, where $b > 0$. How can you determine whether it makes sense to determine its doubling time or whether you should instead compute its half-life? (Hint: How do you determine if the function is growing exponentially or decaying exponentially?)