Tutorial Worksheet, 01/27/2016

Instructions: Please work in groups of 3 or 4 students. Please work with students who will attend the same recitation section. You do turn this worksheet in at the end of class; instead, attendance will be recorded so that you get credit for participating in this activity.

Reminder: Our first exam is next Wednesday, 02/03/2016. You should solve all of the exercises in sections 1.1–1.8 to prepare for this exam. You should also be familiar with the steps involved in formulating and testing a mathematical model.

Solutions to 01/25 Worksheet:

01/25: A Quadratic Model of Mold Growth

Explore 1.5.1. Using the photographs on pp. 22–23, measure the areas of the mold for the days 2 and 6 and enter the values into the table in Figure 1.15. The grid lines are at 2 mm intervals. Check your additional data points with points on the graph.

Solution: The first mold sample (day 2) covers up a rectangular grid of 6 boxes minus parts of the four corners, so an estimate of the area is that it covers 5 boxes in all, which measures

$$5 \times 4 = 20\,\mathrm{mm}^2.$$

(The boxes are two millimeters by two millimeters, so each has an area of four square millimeters.)

The second mold sample (day 6) covers up parts of a rectangular grid of 8 by 8 or 64 boxes. Large portions of the corners are not covered up. There are about 20 boxes which fit into these corners, so an estimate of the area is 44 boxes, which measures

$$44 \times 4 = 172\,\mathrm{mm}^2.$$

Explore 1.5.2. Find real numbers A and B so that $y = AB^t$ approximates the mold growth data in Figure 1.15. You should choose two data points and insist that these satisfy the equation; then solve for A and B. Similarly, find a parabola $y = at^2 + bt + c$ that approximates the mold data. Draw graphs of the mold data, the exponential function, and the parabola on a single set of axes.

Solution: To fit the data with an exponential curve $y = AB^t$, we need to choose two data points (since there are two unknowns, namely A and B). I'll choose $(0, 4)$ and $(5, 126)$ (from Figure 1.15). Here (x, y) represents the day, x, and the area, y. We require that both points lie on the curve. Therefore,

1. $4 = AB^0$ which implies that $4 = A$ since $B = 0$, and
2. $126 = AB^5$ which implies that $126 = 4B^5$ and so $B = \sqrt[5]{126/4} = \sqrt[5]{31.5} \approx 2$

Thus, we expect, perhaps, that $y = 4 \cdot 2^x$ fits the data.

Follow-up Question: Does this curve seem to fit the data?

To fit the data with a quadratic curve $y = ax^2 + bx + c$, we need to choose three data points. I'll choose $(0, 4)$, $(3, 50)$, and $(5, 126)$. We require that all three points lie on the curve. Therefore,

1. $4 = a \cdot 0^2 + b \cdot 0 + c$ and so $c = 4$, and
2. $50 = a \cdot 3^2 + b \cdot 3 + c$ and so $50 = 9a + 3b + 4$, and
3. $126 = a \cdot 5^2 + b \cdot 5 + c$ and so $126 = 25a + 5b + 4$

We can solve for a and b by solving a system of two equations and two unknowns. Please do this now and show your work below. You can use your calculator and round-off since the numbers not especially nice.

Solutions to 01/26 Worksheet:

01/26: Exponential & Logarithmic Functions
Exercises: Write each function in the form $y = cb^x$, where y is the dependent variable and x is the independent variable.

1. $y = (3 \cdot 2^x) \cdot (4 \cdot 2^{x+1})$
   Answer: $y = 24 \cdot 4^x$
2. $y = (3 \cdot 2^x)^3$
   Answer: $y = 27 \cdot 8^x$
3. $y = 2e^x / (4e^{-x})$
   Answer: $y = \frac{1}{2} \cdot (e^2)^x$, so that the base is $b = e^2$.

Exercise: Solve $P_{t+1} - P_t = rP_t$, where $r > 0$ is a constant, and write the solution equation in the form $P_t = AB^t$.

Solution: $P_{t+1} = (1+r)P_t$ which implies that

$$P_t = (1+r)P_{t-1} = (1+r)^2 P_{t-2} = \cdots = (1+r)^t P_0.$$

So, $P_t = P_0 \cdot (1+r)^t$.

Exercises: Use the properties of logarithms to write each function in the form $y = \log_b (cx^r)$, where y is the dependent variable and x is the independent variable.

1. $y = 3\log_2 (2x) + 4\log_2 x$

   Answer: $y = \log_2 (8x^7)$

2. $y = 3\log_{10} x - \log_{10} (3x)$

   Answer: $y = \log_{10} (\frac{1}{3} \cdot x^2)$

Exercises: Determine the doubling time of each of the exponential functions below. Give both an exact and a numerical answer.

1. $y = 2^{3t}$

   Solution: Initial value: $y = 1$; double this is 2. Solve $2 = 2^{3t}$. Therefore, the doubling time is $t = \frac{1}{3}$.

2. $y = 10^{0.1t}$

   Solution: Initial value: $y = 1$; double this is 2. Solve $2 = 10^{0.1t}$. Therefore, the doubling time is $t = 10(\ln 2)/(\ln 10) \approx 3.01$

Exercises: Determine the half-life of each of exponential functions below. Give both and exact and a numerical answer.

1. $y = 100 \cdot (0.8)^t$

   Solution: Initial value is 100; half of this is 50. Solve $50 = 100 \cdot (0.8)^t$. Answer: $t = (\ln (1/2))/(\ln (0.8)) \approx 3.11$.

2. $y = (0.1)^{0.1t}$

   Solution: Initial value is 1; half of this is 0.5. Solve $0.5 = (0.1)^{0.1t}$. Answer: $t = 10 \cdot (\ln 0.5)/(\ln (0.1)) \approx 3.01$.

Follow-up question: It seems perhaps surprising that we obtained nearly the same answer in the second part of each of the previous two exercises. Are these answers exactly the same? Why or why not?

Exercise: If $y = cb^x$, where $b > 0$. How can you determine whether it makes sense to determine its doubling time or whether you should instead compute its half-life? (Hint: How do you determine if the function is growing exponentially or decaying exponentially?)

Solution: If $0 < b < 1$ and $c > 0$, then the curve is a decaying exponential. If $b > 1$ and $c > 0$, then the curve is a growing exponential. (If $c < 0$, the curve are reflected in the x-axis; these curves do not occur in half-life and doubling problems since the y-values are negative.) If the curve is a decaying exponential, you find the half-life. If the curve is a growing exponential, then you find the doubling time. So, it's just a matter of figuring out if the base of the exponential curve is less than one or greater than one. The base of an exponential curve is not allowed to be negative, zero, or one.