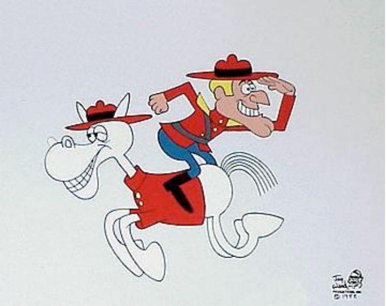


Exotic Cyclic Group Actions on Smooth 4-Manifolds

Ronald Fintushel
Michigan State University
March, 2009

Joint work with **Ron Stern** and **Nathan Sunukjian**

How exotic is 'exotic'?

Exotic smooth structures

Important consequences of Seiberg-Witten (and Donaldson) theory

- Existence of nondiffeomorphic but homeomorphic smooth 4-manifolds
- Existence of surfaces in a fixed smooth 4-manifold which are topologically but not smoothly equivalent

Exotic smooth group actions

- Existence of smooth actions of a group on a smooth 4-manifold which are equivariantly homeomorphic but not equivariantly diffeomorphic.

Example: Exotic involutions on S^4, Quotient = Fake RP^4
(F- Stern/ Cappell - Shaneson, Gompf)

- Want orientation-preserving examples

How exotic is 'exotic'?

Exotic smooth structures

Important consequences of Seiberg-Witten (and Donaldson) theory

• Existence of nondiffeomorphic but homeomorphic smooth 4-manifolds

• Existence of surfaces in a fixed smooth 4-manifold which are topologically but not smoothly equivalent

Exotic smooth group actions

• Existence of smooth actions of a group on a smooth 4-manifold which are equivariantly homeomorphic but not equivariantly diffeomorphic.

Example: Exotic involutions on S^4, Quotient = Fake RP^4

(F- Stern/ Cappell - Shaneson, Gompf)

• Want orientation-preserving examples

How exotic is 'exotic'?

Exotic smooth structures

Important consequences of Seiberg-Witten (and Donaldson) theory

• Existence of nondiffeomorphic but homeomorphic smooth 4-manifolds

• Existence of surfaces in a fixed smooth 4-manifold which are topologically but not smoothly equivalent

Exotic smooth group actions

• Existence of smooth actions of a group on a smooth 4-manifold which are equivariantly homeomorphic but not equivariantly diffeomorphic.
Example: Exotic involutions on S^4, Quotient = Fake RP^4
(F- Stern/ Cappell - Shaneson, Gompf)

• Want orientation-preserving examples

How exotic is ‘exotic’?

Exotic smooth structures

Important consequences of Seiberg-Witten (and Donaldson) theory

- Existence of nondiffeomorphic but homeomorphic smooth 4-manifolds
- Existence of surfaces in a fixed smooth 4-manifold which are topologically but not smoothly equivalent

Exotic smooth group actions

- Existence of smooth actions of a group on a smooth 4-manifold which are equivariantly homeomorphic but not equivariantly diffeomorphic.

Example: Exotic involutions on S^4, Quotient $=$ Fake RP^4
(F- Stern/ Cappell - Shaneson, Gompf)

- Want orientation-preserving examples

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p, \ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y, \ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$$
$$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p,\ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y,\ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$$
$$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p,\ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y,\ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$

$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p, \ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y, \ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$$
$$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p,\ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y,\ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$

$\quad \cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p,\ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y,\ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$

$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p,\ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y,\ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$$
$$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Ue's examples

Ue's Theorem, 1998

For any nontrivial finite group G there exists a smooth 4-manifold that has infinitely many free G-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

Y: $\mathbb{Q}$-homology S^4 with $\pi_1(Y) \to G$, onto, s. t. corr. cover is $\tilde{Y} = S^2 \times S^2 \# Z$, some Z. Get Y by spinning known 3D example.

$X_0 = E(2)_p,\ X_1 = E(2)_q$, $p \neq q$ odd (log transformed K3's)

$X_0 \# Y,\ X_1 \# Y$ homeo not diffeo using Seiberg-Witten

The G-covers Q_i come from $\pi_1(X_i \# Y) \to \pi_1(Y) \to G$

$$Q_i \cong \tilde{Y} \# |G| X_i \cong S^2 \times S^2 \# Z \# |G| X_i$$
$$\cong S^2 \times S^2 \# Z \# |G| X_j \cong Q_j$$

since the $E(2)_p$'s stabilize after one $\# S^2 \times S^2$.

- The Q_i are reducible.

Exotic cyclic group actions

Theorem (F., Stern, Sunukjian)

Let Y be a simply connected 4-manifold with $b^+ \geq 1$ containing an embedded surface Σ of genus $g \geq 1$ of nonnegative self-intersection. Suppose that $\pi_1(Y \smallsetminus \Sigma) = \mathbb{Z}_d$ and that the pair (Y, Σ) has a nontrivial relative Seiberg-Witten invariant. Suppose also that Σ contains a nonseparating loop which bounds an embedded 2-disk in $Y \smallsetminus \Sigma$. Let d' divide d, and let X be the (simply connected) d'-fold cover of Y branched over Σ. Then X admits an infinite family of smoothly distinct but topologically equivalent actions of $\mathbb{Z}_{d'}$.

Some simple examples

Curves in $\mathbb{CP}^2$

$Y = \mathbb{CP}^2$, $\Sigma =$ embedded degree d curve.
$X =$ degree d hypersurface in $\mathbb{CP}^3$

If $d = 3$, $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2 \implies$ we have infinitely many smoothly inequivalent topologically equivalent $\mathbb{Z}_3$-actions on $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$.

If $d = 4$, $X = K3$, $\implies$ smoothly inequivalent topologically equivalent $\mathbb{Z}_4$-actions on the K3-surface.
Also theorem $\implies$ families of $\mathbb{Z}_2$ and $\mathbb{Z}_3$-actions on K3.

$\mathbb{Z}_5$-actions on quintics, etc.

Some simple examples

Curves in $\mathbb{CP}^2$

$Y = \mathbb{CP}^2$, $\Sigma =$ embedded degree d curve.
$X =$ degree d hypersurface in $\mathbb{CP}^3$

If $d = 3$, $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2 \implies$ we have infinitely many smoothly inequivalent topologically equivalent $\mathbb{Z}_3$-actions on $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$.

If $d = 4$, $X = K3$, $\implies$ smoothly inequivalent topologically equivalent $\mathbb{Z}_4$-actions on the K3-surface.
Also theorem $\implies$ families of $\mathbb{Z}_2$ and $\mathbb{Z}_3$-actions on K3.

$\mathbb{Z}_5$-actions on quintics, etc.

Knot and rim surgery (F. - Stern)

Knot surgery

K: Knot in S^3, T: square 0 essential torus in X

$X_K = X \smallsetminus N_T \cup S^1 \times (S^3 \smallsetminus N_K)$

$S^1 \times (S^3 \smallsetminus N_K)$ has the homology of $T^2 \times D^2$.

Facts

- If X and $X \smallsetminus T$ both simply connected, so is X_K. (So X_K homeo to X)
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Rim surgery

$\Sigma \subset X$: embedded orientable surface in simply connected 4-manifold.

C: homologically essential loop in Σ

Rim torus: preimage of C in bdry of normal bundle of Σ.

Rim surgery = knot surgery on rim torus.

Can change embedding type of Σ. Get $\Sigma_K \subset X$.

Knot and rim surgery (F. - Stern)

Knot surgery

K: Knot in S^3, T: square 0 essential torus in X

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

$S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts

- If X and $X \setminus T$ both simply connected, so is X_K. (So X_K homeo to X)
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Rim surgery

$\Sigma \subset X$: embedded orientable surface in simply connected 4-manifold.

C: homologically essential loop in Σ

Rim torus: preimage of C in bdry of normal bundle of Σ.

Rim surgery = knot surgery on rim torus.

Can change embedding type of Σ. Get $\Sigma_K \subset X$.

Knot and rim surgery (F. - Stern)

Knot surgery

K: Knot in S^3, T: square 0 essential torus in X

$X_K = X \smallsetminus N_T \cup S^1 \times (S^3 \smallsetminus N_K)$

$S^1 \times (S^3 \smallsetminus N_K)$ has the homology of $T^2 \times D^2$.

Facts

- If X and $X \smallsetminus T$ both simply connected, so is X_K. (So X_K homeo to X)
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Rim surgery

$\Sigma \subset X$: embedded orientable surface in simply connected 4-manifold.

C: homologically essential loop in Σ

Rim torus: preimage of C in bdry of normal bundle of Σ.

Rim surgery = knot surgery on rim torus.

Can change embedding type of Σ. Get $\Sigma_K \subset X$.

Knot and rim surgery (F. - Stern)

Knot surgery

K: Knot in S^3, T: square 0 essential torus in X

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

$S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts

- If X and $X \setminus T$ both simply connected, so is X_K. (So X_K homeo to X)
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Rim surgery

$\Sigma \subset X$: embedded orientable surface in simply connected 4-manifold.

C: homologically essential loop in Σ

Rim torus: preimage of C in bdry of normal bundle of Σ.

Rim surgery = knot surgery on rim torus.

Can change embedding type of Σ. Get $\Sigma_K \subset X$.

Knot and rim surgery (F. - Stern)

Knot surgery

K: Knot in S^3, T: square 0 essential torus in X

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

$S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts

- If X and $X \setminus T$ both simply connected, so is X_K. (So X_K homeo to X)
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Rim surgery

$\Sigma \subset X$: embedded orientable surface in simply connected 4-manifold.

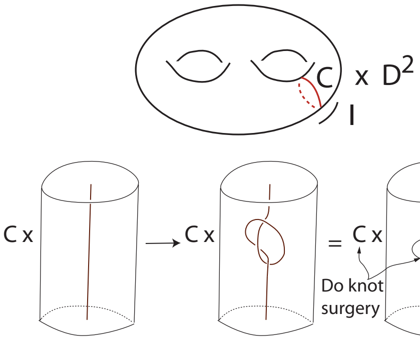
C: homologically essential loop in Σ

Rim torus: preimage of C in bdry of normal bundle of Σ.

Rim surgery = knot surgery on rim torus.

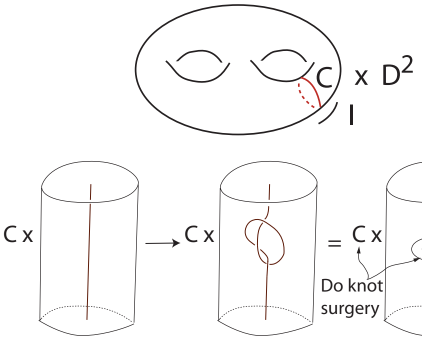
Can change embedding type of Σ. Get $\Sigma_K \subset X$.

More on rim surgery



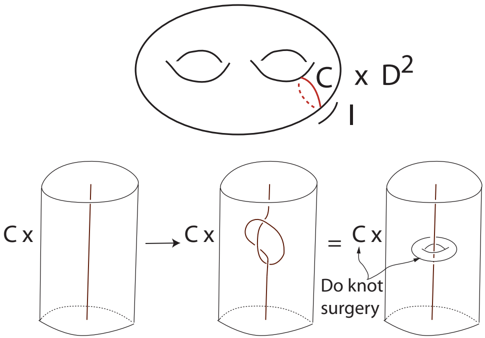
Spinning a knot K in S^3 gives 2-knot in S^4:
S^1-action on S^4. Orbit space B^3.
Spun knot = preimage of knotted arc. Preimage of ∂B^3= twin
Knot surgery replaces $C \times S^1 \times D^2$ with $S^4 \setminus$(spun knot $\cup$ twin)
$C \times B^3$ = complement of trivial twin in S^4.

More on rim surgery



Spinning a knot K in S^3 gives 2-knot in S^4:
S^1-action on S^4. Orbit space B^3.
Spun knot $=$ preimage of knotted arc. Preimage of $\partial B^3 =$ twin
Knot surgery replaces $C \times S^1 \times D^2$ with $S^4 \setminus$(spun knot $\cup$ twin)
$C \times B^3 =$ complement of trivial twin in S^4.

More on rim surgery



Spinning a knot K in S^3 gives 2-knot in S^4:
S^1-action on S^4. Orbit space B^3.
Spun knot $=$ preimage of knotted arc. Preimage of $\partial B^3 =$ twin
Knot surgery replaces $C \times S^1 \times D^2$ with $S^4 \setminus$(spun knot $\cup$ twin)
$C \times B^3$ $=$ complement of trivial twin in S^4.

(Can't get enough of that) Rim surgery

Theorem (F - Stern). Let $g(\Sigma) > 0$. If $\pi_1(X) = 0 = \pi_1(X \setminus \Sigma)$ then there is a self-homeo of X throwing Σ_K on Σ. If $\Sigma^2 > 0$, then the relative SW-invariant of (X, Σ_K) is the relative SW-invariant of (X, Σ) times the Alexander polynomial of K.

Get smoothly inequivalent embeddings if original SW inv't is $\neq 0$. (E.g. symplectic submanifold.)

Relative SW-invariant lives in monopole Floer homology group.

Want to take cyclic branched covers — need $\pi_1(X \setminus \Sigma) = \mathbb{Z}_d$.
Problem: Rim surgery will not preserve this condition.

Solution (Kim - Ruberman) k-Twist-spun rim surgery does preserve $\pi_1 = \mathbb{Z}_d$ as long as k is prime to d.
In fact, they show that the new surface obtained is topologically equivalent to the old one in this case.
Relative SW-invariant is the same as for ordinary rim surgery.

(Can't get enough of that) Rim surgery

Theorem (F - Stern). Let $g(\Sigma) > 0$. If $\pi_1(X) = 0 = \pi_1(X \setminus \Sigma)$ then there is a self-homeo of X throwing Σ_K on Σ. If $\Sigma^2 > 0$, then the relative SW-invariant of (X, Σ_K) is the relative SW-invariant of (X, Σ) times the Alexander polynomial of K.

Get smoothly inequivalent embeddings if original SW inv't is $\neq 0$.
(E.g. symplectic submanifold.)
Relative SW-invariant lives in monopole Floer homology group.

Want to take cyclic branched covers — need $\pi_1(X \setminus \Sigma) = \mathbb{Z}_d$.
Problem: Rim surgery will not preserve this condition.

Solution (Kim - Ruberman) k-Twist-spun rim surgery does preserve $\pi_1 = \mathbb{Z}_d$ as long as k is prime to d.
In fact, they show that the new surface obtained is topologically equivalent to the old one in this case.
Relative SW-invariant is the same as for ordinary rim surgery.

(Can't get enough of that) Rim surgery

Theorem (F - Stern). Let $g(\Sigma) > 0$. If $\pi_1(X) = 0 = \pi_1(X \setminus \Sigma)$ then there is a self-homeo of X throwing Σ_K on Σ. If $\Sigma^2 > 0$, then the relative SW-invariant of (X, Σ_K) is the relative SW-invariant of (X, Σ) times the Alexander polynomial of K.

Get smoothly inequivalent embeddings if original SW inv't is $\neq 0$.
(E.g. symplectic submanifold.)
Relative SW-invariant lives in monopole Floer homology group.

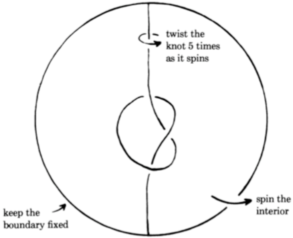
Want to take cyclic branched covers — need $\pi_1(X \setminus \Sigma) = \mathbb{Z}_d$.
Problem: Rim surgery will not preserve this condition.

Solution (Kim - Ruberman) k-Twist-spun rim surgery does preserve $\pi_1 = \mathbb{Z}_d$ as long as k is prime to d.
In fact, they show that the new surface obtained is topologically equivalent to the old one in this case.
Relative SW-invariant is the same as for ordinary rim surgery.

Twist-spinning

Twist-spinning a knot

K: knot in S^3. Twist-spinning operation due to Zeeman.
Get knotted S^2 in S^4 and circle action.



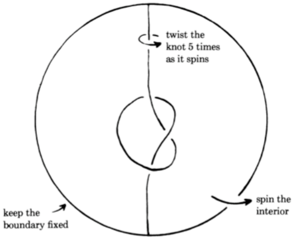
Twist-spun rim surgery, $\Sigma_{K,k}$

Twist-rim $C \times S^1 \times D^2$ with S^4(twist-spun knot $\cup$ twin)
$C \times I \times D^2$ replaced by complement of trivial twin in S^4.
Annulus on surface replaced by twist-spun knot minus polar caps.

Twist-spinning

Twist-spinning a knot

K: knot in S^3. Twist-spinning operation due to Zeeman.
Get knotted S^2 in S^4 and circle action.



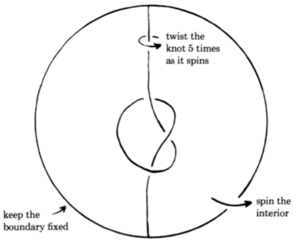
Twist-spun rim surgery, $\Sigma_{K,k}$

Twist-rim $C \times S^1 \times D^2$ with S^4(twist-spun knot $\cup$ twin)
$C \times I \times D^2$ replaced by complement of trivial twin in S^4.
Annulus on surface replaced by twist-spun knot minus polar caps.

Twist-spinning

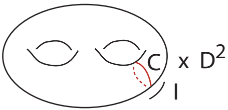
Twist-spinning a knot

K: knot in S^3. Twist-spinning operation due to Zeeman.
Get knotted S^2 in S^4 and circle action.



Twist-spun rim surgery, $\Sigma_{K,k}$

Twist-rim $C \times S^1 \times D^2$ with S^4(twist-spun knot $\cup$ twin)
$C \times I \times D^2$ replaced by complement of trivial twin in S^4.
Annulus on surface replaced by twist-spun knot minus polar caps.



Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set = S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.

k-twist spin of K = $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$

gets replaced with $S^4 \setminus \mathrm{Nbd}(p^{-1}(E_k))$ where
E_k = closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set = S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.

k-twist spin of K = $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$

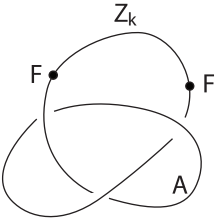
gets replaced with $S^4 \setminus \mathrm{Nbd}(p^{-1}(E_k))$ where E_k = closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set $=$ S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



k-twist spin of K $=$ $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$

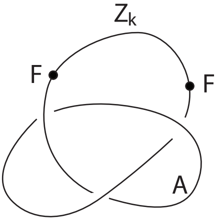
gets replaced with $S^4 \setminus \text{Nbd}(p^{-1}(E_k))$ where $E_k =$ closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set $=$ S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



k-twist spin of K $=$ $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$

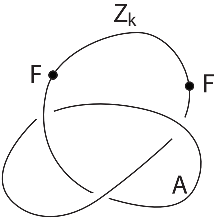
gets replaced with $S^4 \setminus \text{Nbd}(p^{-1}(E_k))$ where
E_k = closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set $=$ S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



k-twist spin of K $=$ $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$

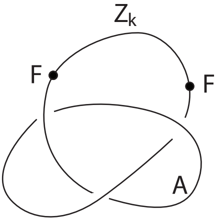
gets replaced with $S^4 \setminus \mathrm{Nbd}(p^{-1}(E_k))$ where $E_k =$ closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set $=$ S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

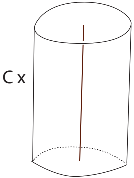
Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



k-twist spin of K $=$ $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$



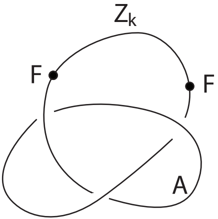
gets replaced with $S^4 \setminus \mathrm{Nbd}(p^{-1}(E_k))$ where E_k = closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set $=$ S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

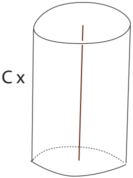
Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



k-twist spin of K $=$ $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$



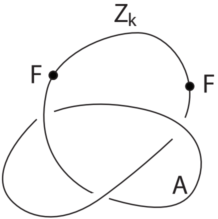
gets replaced with $S^4 \setminus \mathrm{Nbd}(p^{-1}(E_k))$ where E_k = closed arc labeled '$\mathbb{Z}_k$'.

Circle actions on S^4 and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: B^3 or S^3
Fixed point set = S^0 or S^2. Exceptional orbit image $0, 1$ or 2 arcs.

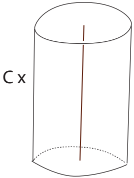
Twist-spinning a knot

K: knot in S^3. S^1-action on S^4 with orbit space S^3, $p : S^4 \to S^3$ where the isotropy type corresponding to the arc A is trivial.



k-twist spin of K = $p^{-1}(\bar{A}) \subset S^4$.

Twist-spun rim surgery, $\Sigma_{K,k}$



gets replaced with $S^4 \setminus \text{Nbd}(p^{-1}(E_k))$ where E_k = closed arc labeled '$\mathbb{Z}_k$'.

Relative Seiberg-Witten invariants

By blowing up, assume $\Sigma \cdot \Sigma = 0$.

Seiberg-Witten invariant of $Y \smallsetminus N(\Sigma)$ obtained from spin^c-structures $\mathfrak{s}$ on Y satisfying $\langle c_1(\mathfrak{s}), \Sigma \rangle = 2g - 2$.

$SW_{(Y|\Sigma)} : H_2(Y \smallsetminus N(\Sigma), \Sigma \times S^1; \mathbb{R}) \to \mathbb{R}$ (Kronheimer/ Mrowka)

Role of basic classes played by $z \in \pi_0(\mathcal{B}(Y \smallsetminus N(\Sigma); [a_0]))$, principal homogeneous space for $H^2(Y \smallsetminus N(\Sigma), \partial)$
$z = [(A, \Phi)]$ solving SW eq'ns.
a_0: unique spin^c-structure on $\Sigma \times S^1$ of degree $2g - 2$

Knot surgery theorem

Basic classes for $Y|\Sigma_{K,k}$: $z + j\rho$, $\rho = \mathrm{PD}(\text{rim torus})$, t^j has $\neq 0$ coeff in $\Delta_K(t)$. $\Longrightarrow$
Given (Y, Σ, C) there is an infinite family of knots K and surfaces $\Sigma_{K,k}$ all topologically equivalent but smoothly inequivalent obtained by (K, k)-twist-rim surgery.

Relative Seiberg-Witten invariants

By blowing up, assume $\Sigma \cdot \Sigma = 0$.

Seiberg-Witten invariant of $Y \setminus N(\Sigma)$ obtained from spin^c-structures $\mathfrak{s}$ on Y satisfying $\langle c_1(\mathfrak{s}), \Sigma \rangle = 2g - 2$.

$SW_{(Y|\Sigma)} : H_2(Y \setminus N(\Sigma), \Sigma \times S^1; \mathbb{R}) \to \mathbb{R}$ (Kronheimer/ Mrowka)

Role of basic classes played by $z \in \pi_0(\mathcal{B}(Y \setminus N(\Sigma); [a_0]))$, principal homogeneous space for $H^2(Y \setminus N(\Sigma), \partial)$

$z = [(A, \Phi)]$ solving SW eq'ns.

a_0: unique spin^c-structure on $\Sigma \times S^1$ of degree $2g - 2$

Knot surgery theorem

Basic classes for $Y|\Sigma_{K,k}$: $z + j\rho$, $\rho = \mathrm{PD}(\text{rim torus})$, t^j has $\neq 0$ coeff in $\Delta_K(t)$. $\implies$

Given (Y, Σ, C) there is an infinite family of knots K and surfaces $\Sigma_{K,k}$ all topologically equivalent but smoothly inequivalent obtained by (K, k)-twist-rim surgery.

Relative Seiberg-Witten invariants

By blowing up, assume $\Sigma \cdot \Sigma = 0$.

Seiberg-Witten invariant of $Y \setminus N(\Sigma)$ obtained from spin^c-structures $\mathfrak{s}$ on Y satisfying $\langle c_1(\mathfrak{s}), \Sigma \rangle = 2g - 2$.

$SW_{(Y|\Sigma)} : H_2(Y \setminus N(\Sigma), \Sigma \times S^1; \mathbb{R}) \to \mathbb{R}$ (Kronheimer/ Mrowka)

Role of basic classes played by $z \in \pi_0(\mathcal{B}(Y \setminus N(\Sigma); [a_0]))$, principal homogeneous space for $H^2(Y \setminus N(\Sigma), \partial)$
$z = [(A, \Phi)]$ solving SW eq'ns.
a_0: unique spin^c-structure on $\Sigma \times S^1$ of degree $2g - 2$

Knot surgery theorem

Basic classes for $Y|\Sigma_{K,k}$: $z + j\rho$, $\rho = \text{PD}(\text{rim torus})$, t^j has $\neq 0$ coeff in $\Delta_K(t)$. $\implies$
Given (Y, Σ, C) there is an infinite family of knots K and surfaces $\Sigma_{K,k}$ all topologically equivalent but smoothly inequivalent obtained by (K, k)-twist-rim surgery.

Relative Seiberg-Witten invariants

By blowing up, assume $\Sigma \cdot \Sigma = 0$.

Seiberg-Witten invariant of $Y \smallsetminus N(\Sigma)$ obtained from spin^c-structures $\mathfrak{s}$ on Y satisfying $\langle c_1(\mathfrak{s}), \Sigma \rangle = 2g - 2$.

$SW_{(Y|\Sigma)} : H_2(Y \smallsetminus N(\Sigma), \Sigma \times S^1; \mathbb{R}) \to \mathbb{R}$ (Kronheimer/ Mrowka)

Role of basic classes played by $z \in \pi_0(\mathcal{B}(Y \smallsetminus N(\Sigma); [a_0]))$, principal homogeneous space for $H^2(Y \smallsetminus N(\Sigma), \partial)$

$z = [(A, \Phi)]$ solving SW eq'ns.

a_0: unique spin^c-structure on $\Sigma \times S^1$ of degree $2g - 2$

Knot surgery theorem

Basic classes for $Y|\Sigma_{K,k}$: $z + j\rho$, $\rho =$ PD(rim torus), t^j has $\neq 0$ coeff in $\Delta_K(t)$. $\Longrightarrow$

Given (Y, Σ, C) there is an infinite family of knots K and surfaces $\Sigma_{K,k}$ all topologically equivalent but smoothly inequivalent obtained by (K, k)-twist-rim surgery.

Relative Seiberg-Witten invariants

By blowing up, assume $\Sigma \cdot \Sigma = 0$.

Seiberg-Witten invariant of $Y \smallsetminus N(\Sigma)$ obtained from spin^c-structures $\mathfrak{s}$ on Y satisfying $\langle c_1(\mathfrak{s}), \Sigma\rangle = 2g - 2$.

$SW_{(Y|\Sigma)} : H_2(Y \smallsetminus N(\Sigma), \Sigma \times S^1; \mathbb{R}) \to \mathbb{R}$ (Kronheimer/ Mrowka)

Role of basic classes played by $z \in \pi_0(\mathcal{B}(Y \smallsetminus N(\Sigma); [a_0]))$, principal homogeneous space for $H^2(Y \smallsetminus N(\Sigma), \partial)$
$z = [(A, \Phi)]$ solving SW eq'ns.
a_0: unique spin^c-structure on $\Sigma \times S^1$ of degree $2g - 2$

Knot surgery theorem

Basic classes for $Y|\Sigma_{K,k}$: $z + j\rho$, $\rho = \mathrm{PD}(\text{rim torus})$, t^j has $\neq 0$ coeff in $\Delta_K(t)$. $\Longrightarrow$
Given (Y, Σ, C) there is an infinite family of knots K and surfaces $\Sigma_{K,k}$ all topologically equivalent but smoothly inequivalent obtained by (K, k)-twist-rim surgery.

Cyclic group actions

Y: simply connected smooth 4-manifold.
Σ genus ≥ 1 surface embedded in Y such that $\pi_1(Y \setminus \Sigma) = \mathbb{Z}_d$.
C: nonseparating loop on Σ, bounds D^2 in complement.
$X = d$-fold branched cyclic cover.

Choose k relatively prime to d $\exists$ family of knots K_i so that d-fold branched covers X_i of $(Y, \Sigma_{K_i,k})$ are all topologically equivalent but smoothly distinct covers.

$\Longrightarrow$

Topologically equivalent but smoothly distinct actions of $\mathbb{Z}_d$.

Need to see that X_i are diffeomorphic to each other.

Cyclic group actions

Y: simply connected smooth 4-manifold.
Σ genus ≥ 1 surface embedded in Y such that $\pi_1(Y \setminus \Sigma) = \mathbb{Z}_d$.
C: nonseparating loop on Σ, bounds D^2 in complement.
$X = d$-fold branched cyclic cover.

Choose k relatively prime to d $\exists$ family of knots K_i so that d-fold branched covers X_i of $(Y, \Sigma_{K_i,k})$ are all topologically equivalent but smoothly distinct covers.

$\Longrightarrow$
Topologically equivalent but smoothly distinct actions of $\mathbb{Z}_d$.

Need to see that X_i are diffeomorphic to each other.

Cyclic group actions

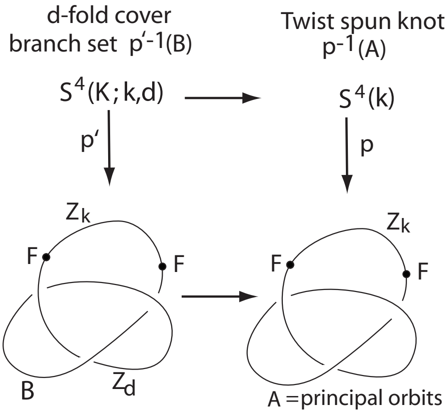
Y: simply connected smooth 4-manifold.
Σ genus ≥ 1 surface embedded in Y such that $\pi_1(Y \setminus \Sigma) = \mathbb{Z}_d$.
C: nonseparating loop on Σ, bounds D^2 in complement.
X = d-fold branched cyclic cover.

Choose k relatively prime to d $\exists$ family of knots K_i so that d-fold branched covers X_i of $(Y, \Sigma_{K_i,k})$ are all topologically equivalent but smoothly distinct covers.

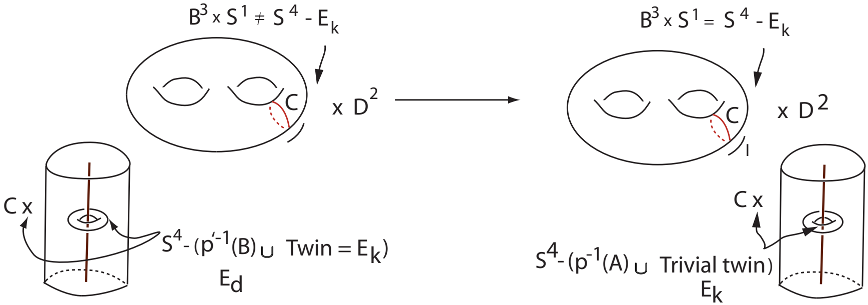
$\Longrightarrow$
Topologically equivalent but smoothly distinct actions of $\mathbb{Z}_d$.

Need to see that X_i are diffeomorphic to each other.

Branched covers of twist-spun knots



Covering spaces



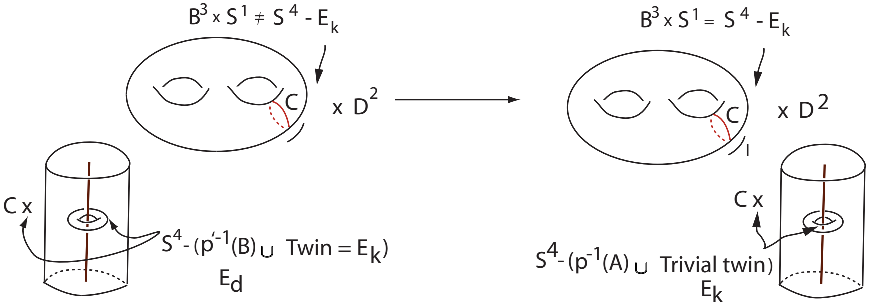
In cover, replacing $C \times I \times D^2$ with $S^4 \setminus E_k \neq S^1 \times B^3$

C bounds disk, $C \times I \times D^2 \cup \text{Nbd(disk)} = B^4$ in X

After knot surgery in Y, B^4 in cover becomes $S^4(K; k, d) \setminus B^4$.
$\Longrightarrow$

X_K diffeomorphic to X □

Covering spaces



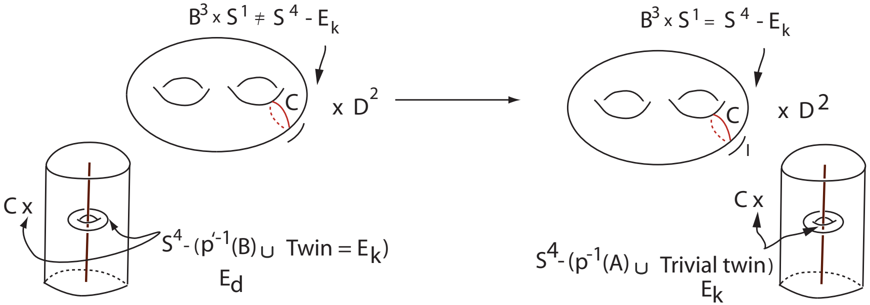
In cover, replacing $C \times I \times D^2$ with $S^4 \setminus E_k \neq S^1 \times B^3$

C bounds disk, $C \times I \times D^2 \cup \mathrm{Nbd(disk)} = B^4$ in X

After knot surgery in Y, B^4 in cover becomes $S^4(K; k, d) \setminus B^4$.
$\Longrightarrow$

X_K diffeomorphic to X □

Covering spaces



In cover, replacing $C \times I \times D^2$ with $S^4 \setminus E_k \neq S^1 \times B^3$

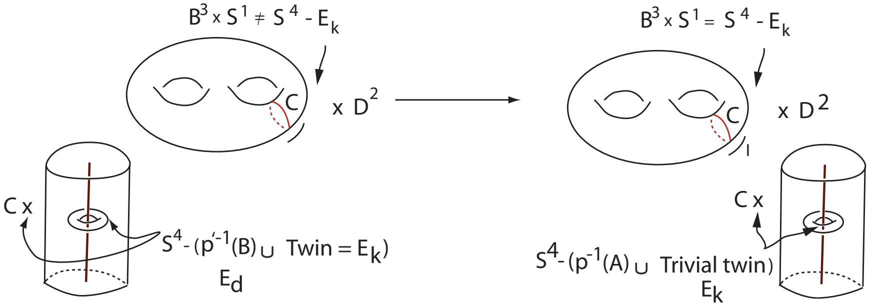
C bounds disk, $C \times I \times D^2 \cup \text{Nbd(disk)} = B^4$ in X

After knot surgery in Y, B^4 in cover becomes $S^4(K; k, d) \setminus B^4$.

$\Longrightarrow$

X_K diffeomorphic to X $\square$

Covering spaces



In cover, replacing $C \times I \times D^2$ with $S^4 \setminus E_k \neq S^1 \times B^3$

C bounds disk, $C \times I \times D^2 \cup \mathrm{Nbd(disk)} = B^4$ in X

After knot surgery in Y, B^4 in cover becomes $S^4(K; k, d) \setminus B^4$.

$\Longrightarrow$

$$X_K \text{ diffeomorphic to } X \qquad \square$$