CHAPTER XI : THE KOSZUL COMPLEX

§1: REVIEW OF EXTERIOR ALGEBRA

Let A be a ring and M an A-module. Consider A as a graded ring by the trivial grading (i.e. $A = \bigoplus_{n=0}^{\infty} A_n$ where $A = A_0$ and $A_n = 0$ for all $n > 0$). Let $M^{\otimes i}$ denote the ith tensor power of M, i.e. $M^{\otimes i} = M \otimes \cdots \otimes M$ with i factors of M if $i > 0$ and $M^{\otimes 0} = A$. Then $\otimes M = \bigoplus_{i=0}^{\infty} M^{\otimes i}$ is a graded A-module. The bilinear map $M^{\otimes i} \times M^{\otimes j} \longrightarrow M^{\otimes i+j}$ induced by $(x_1 \otimes \cdots \otimes x_i, y_1 \otimes \cdots \otimes y_j) \mapsto x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j$ extends linearly to a multiplication on $\otimes M$. With this definition $\otimes M$ becomes a graded associative A-algebra which is not commutative in general. $\otimes M$ is called the <u>tensor algebra</u> of M. The tensor algebra is characterized by a

(11.1) <u>Universal property</u>: Let B be an A-algebra (not necessarily commutative) and $\varphi : M \longrightarrow B$ an A-linear map. Then there is a unique A-algebra homomorphism $\psi : \otimes M \longrightarrow B$ extending φ, i.e. $\psi|_{M^{\otimes 1} = M} = \varphi$.

The <u>exterior algebra</u> ΛM is the residue class algebra $\Lambda M = \otimes M / \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $\{x \otimes x \mid x \in M\}$. Since $\mathcal{I}$ is generated by homogeneous elements, ΛM is a graded A-algebra. The product in ΛM is denoted $x \wedge y$. In general ΛM is not commutative; it is alternating: if $x, y \in \Lambda M$ homogeneous, then $x \wedge y = (-1)^{(\deg x)(\deg y)} y \wedge x$ and $x \wedge x = 0$ for x homogeneous and $\deg x$ odd. (Proof: consider $(x+y) \wedge (x+y)$)

(11.2) <u>Universal property</u>: Let B be an A-algebra (not necessarily commutative) and $\varphi : M \longrightarrow B$ an A-linear map with $\varphi(x)^2 = 0$ for all $x \in M$. Then there is a unique A-algebra homomorphism $\psi : \Lambda M \longrightarrow B$ extending φ, i.e. $\psi|_{\Lambda^1 M = M} = \varphi$.

(11.3) <u>Remark</u>: (a) The ith graded component of ΛM is denoted by $\Lambda^i M$ and is called the <u>ith exterior power</u> of M. Obviously, $\Lambda^0 M = A$, $\Lambda^1 M = M$, and for $i \geq 2$ $\Lambda^i M = M^{\otimes i}/(x_1 \otimes \dots \otimes x_i \mid x_j = x_k \text{ for some } j \neq k)$.

(b) Let $x_1, \dots, x_n \in M$. If τ is a permutation of $\{1, \dots, n\}$ then $x_{\tau(1)} \wedge \dots \wedge x_{\tau(n)} = \operatorname{sgn}(\tau)\, x_1 \wedge \dots \wedge x_n$. If $I \subseteq \{1, \dots, n\}$ set $x_I = x_{\nu_1} \wedge \dots \wedge x_{\nu_i}$ if $I = \{\nu_1, \dots, \nu_i\}$ with $\nu_1 < \dots < \nu_i$. If $I, J \subseteq \{1, \dots, n\}$ with $I \cap J = \emptyset$ set $\operatorname{sgn}(I,J) = (-1)^\ell$ where ℓ is the number of $(i,j) \in I \times J$ with $i > j$; if $I \cap J \neq \emptyset$ set $\operatorname{sgn}(I,J) = 0$. Then $x_I \wedge x_J = \operatorname{sgn}(I,J)\, x_{I \cup J}$.

(c) Let $\{x_g\}_{g \in G}$ be a system of generators of M. Then $\Lambda^n M$ is generated by exterior products x_I with $I \subseteq G$ and $|I| = n$. In particular, if $|G| = m < \infty$, then $\Lambda^i M = 0$ for all $i > m$.

(11.4) <u>Remark</u>: Let $\varphi: M \longrightarrow N$ be an A-linear map.

(a) There is a unique A-algebra homomorphism $\Lambda\varphi: \Lambda M \longrightarrow \Lambda N$ so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle \text{nat}}\downarrow & & \downarrow{\scriptstyle \text{nat}} \\ \Lambda M & \xrightarrow{\Lambda\varphi} & \Lambda N \end{array}$$

commutes. $\Lambda\varphi$ is homogeneous of degree 0 with $\Lambda\varphi(x_1 \wedge \dots \wedge x_n) = \varphi(x_1) \wedge \dots \wedge \varphi(x_n)$. (This follows immediately from the universal property of the exterior product).

(b) For all $i > 0$ the sequence $\Lambda^{i-1} M \otimes \ker\varphi \longrightarrow \Lambda^i M \longrightarrow \Lambda^i N \longrightarrow 0$ is exact. In particular, $\ker \Lambda\varphi$ is generated by $\ker\varphi$ (without proof).

(11.5) <u>Remark</u>: Let $\varphi: A \longrightarrow B$ be a homorphism of rings and M an A-module. Then there is a natural isomorphism of graded B-algebras: $(\Lambda M) \otimes_A B \cong \Lambda(M \otimes_A B)$.

Let M and N be A-modules. Define

(a) a grading on $(\Lambda M) \otimes_A (\Lambda N)$ by $[(\Lambda M) \otimes_A (\Lambda N)]_n = \bigoplus_{i+j=n} (\Lambda^i M) \otimes_A (\Lambda^j N)$.

(b) a multiplication on $(\Lambda M) \otimes_A (\Lambda N)$ by $(x \otimes y)(x' \otimes y') = (-1)^{(\deg y)(\deg x')} (x \wedge x') \otimes (y \wedge y')$

for all homogeneous elements $x, x' \in \Lambda M$; $y, y' \in \Lambda N$. Then $(\Lambda M) \otimes (\Lambda N)$ is an alternating graded A-algebra with degree 1 component $(M \otimes A) \oplus (A \otimes N) \cong M \oplus N$. The natural map $M \oplus N \longrightarrow (\Lambda M) \otimes (\Lambda N)$ extends to an A-algebra homomorphism: $\varphi : \Lambda(M \oplus N) \longrightarrow (\Lambda M) \otimes (\Lambda N)$. Conversely, the natural maps $M \hookrightarrow \Lambda(M \oplus N)$ and $N \hookrightarrow \Lambda(M \oplus N)$ extend to homomorphisms of A-algebras $\psi_1 : \Lambda M \longrightarrow \Lambda(M \oplus N)$ and $\psi_2 : \Lambda N \longrightarrow \Lambda(M \oplus N)$. By the universal property of tensor products ψ_1 and ψ_2 induce an A-algebra homomorphism $\psi : (\Lambda M) \otimes (\Lambda N) \longrightarrow \Lambda(M \oplus N)$. Since $[\psi \circ \varphi]_1 = id_{M \oplus N} = [\varphi \circ \psi]_1$ by linear extension ψ and φ are inverse to each other. Thus:

(11.6) Proposition: $(\Lambda M) \otimes_A (\Lambda N) \cong \Lambda(M \oplus N)$ as alternating graded A-algebras.

(11.7) Proposition: (a) If $\{x_1, \dots, x_n\}$ is a generating set of M, then $\{x_I \mid |I| = i\}$ is a generating set of $\Lambda^i M$.

(b) If $\{e_1, \dots, e_n\}$ is a basis of a free module F, then $\{e_I \mid |I| = i\}$ is a basis of $\Lambda^i F$. In particular, $\Lambda^i F$ is free of rank $\binom{n}{i}$.

Proof: (b) Notice that $\Lambda A e_n = A \oplus A e_n$ where $A e_n \cong A$. By (11.6) $\Lambda F = \Lambda(Ae_1 \oplus \cdots \oplus Ae_{n-1}) \otimes \Lambda A e_n$. Thus by induction on n, ΛF has an A-basis $\{e_I, e_I \wedge e_n \mid I \subseteq \{1, \dots, n-1\}\}$.

§2. BASIC PROPERTIES OF THE KOSZUL COMPLEX

Let A be a ring, L an A-module, and $f: L \longrightarrow A$ an A-linear map. The map $\tilde{f}^{(n)}: L^n \longrightarrow \Lambda^{n-1} L$ defined by $\tilde{f}^{(n)}(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{i+1} f(x_i) x_1 \wedge \ldots \hat{x}_i \wedge \ldots \wedge x_n$ is an alternating n-linear map. Thus $\tilde{f}^{(n)}$ factors through an A-linear map $d_f^{(n)}: \Lambda^n L \longrightarrow \Lambda^{n-1} L$ with

$$d_f^{(n)}(x_1 \wedge \ldots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} f(x_i) x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_n$$

for all $x_1, \ldots, x_n \in L$. The collection of all maps $d_f^{(n)}$ defines a graded A-homomorphism $d_f: \Lambda L \longrightarrow \Lambda L$.

(11.8) Remark: (a) d_f has the following properties:

(i) $d_f \circ d_f = 0$

(ii) For all homogeneous $x, y \in \Lambda L$: $d_f(x \wedge y) = d_f(x) \wedge y + (-1)^{\deg x} x \wedge d_f(y)$.

(b) Since $d_f \circ d_f = 0$ we obtain a complex:

$$(*) \quad \ldots \longrightarrow \Lambda^n L \xrightarrow{d_f} \Lambda^{n-1} L \xrightarrow{d_f} \ldots \longrightarrow \Lambda^2 L \xrightarrow{d_f} L \xrightarrow{f} A \longrightarrow 0$$

and (ii) implies that d_f is an antiderivation (of degree -1).

(11.9) Definition: $(*)$ is the Koszul complex of f, denoted $K_\bullet(f)$. If M is an A-module, then $K_\bullet(f, M)$ is the complex $K_\bullet(f) \otimes_A M$, called the Koszul complex of f with coefficients in M. Its differential is denoted by $d_{f,M}$.

(11.10) Proposition: Let A be a ring, L an A-module, and $f: L \longrightarrow A$ an A-linear map.

(a) The Koszul complex $K_\bullet(f)$ carries the structure of an associative graded alternating algebra, namely that of ΛL.

(b) Its differential d_f is an antiderivation of degree -1.

(c) For every A-module M the complex $K_\bullet(f, M)$ is a $K_\bullet(f)$-module in a natural way.

(d) One has $d_{f,M}(x \cdot y) = d_f(x) \cdot y + (-1)^{\deg x} x \cdot d_{f,M}(y)$ for all homogeneous elements x of $K_\bullet(f)$ and all elements $y \in K_\bullet(f, M)$.

Proof: (a) and (b) follow from the definition.
(c) clear, if B is an A-algebra and M an A-module, then $B\otimes_A M$ is a (left) B-module in the natural way.
(d) It is enough to show the claim for $y = w\otimes z$ with $w\in K_\bullet(f)$, $z\in M$. Then $d_{f,M}(x.w\otimes z) = d_{f,M}((x\wedge y)\otimes z) = d_f(x\wedge y)\otimes z$ and the rest follows since d_f is an antiderivation.

Set $Z_\bullet(f) = \ker d_f$, $Z_\bullet(f,M) = \ker d_{f,M}$, and $B_\bullet(f) = \operatorname{im} d_f$, $B_\bullet(f,M) = \operatorname{im} d_{f,M}$.

(11.11) Definition: The homology $H_\bullet(f) = Z_\bullet(f)/B_\bullet(f)$ is the Koszul homology of f. For every A-module M the homology $Z_\bullet(f,M)/B_\bullet(f,M)$ is denoted by $H_\bullet(f,M)$ and called the Koszul homology of f with coefficients in M.

For a subset $S\subseteq K_\bullet(f)$ and a subset $U\subseteq K_\bullet(f,M)$ let $S.U$ denote the A-submodule of $K_\bullet(f,M)$ generated by $\{s.u \mid s\in S, u\in U\}$. Then: $Z_\bullet(f).Z_\bullet(f,M)\subseteq Z_\bullet(f,M)$, $Z_\bullet(f).B_\bullet(f,M)\subseteq B_\bullet(f,M)$ and $B_\bullet(f).Z(f,M)\subseteq B_\bullet(f,M)$. Notice that $K_\bullet(f)\cong K_\bullet(f,A)$. The first inclusion shows that $Z_\bullet(f)$ is a graded A-subalgebra of $K_\bullet(f)$. The second and third inclusions show that $B_\bullet(f)$ is a twosided ideal in $Z_\bullet(f)$.

(11.12) Proposition: Let A be a ring, L an A-module, and $f: L\longrightarrow A$ an A-linear map.
(a) The Koszul homology $H_\bullet(f)$ carries the structure of an associative graded alternating A-algebra.
(b) For every A-module M the homology $H_\bullet(f,M)$ is an $H_\bullet(f)$-module in a natural way.

Proof: (a) $H_\bullet(f)$ is an A-algebra since $Z_\bullet(f)$ is an A-algebra and $B_\bullet(f)$ is an ideal

ideal of $Z_\bullet(f)$. $Z_\bullet(f)$ is an associative graded alternating A-algebra and $B_\bullet(f)$ is homogeneous.

(b) Since $Z_\bullet(f) \cdot Z_\bullet(f,M) \subseteq Z_\bullet(f,M)$, $Z_\bullet(f,M)$ is a $Z_\bullet(f)$-module. Since $Z_\bullet(f) B_\bullet(f,M) \subseteq B_\bullet(f,M)$, $B_\bullet(f,M)$ is a $Z_\bullet(f)$-submodule of $Z_\bullet(f,M)$ and since $B_\bullet(f) Z_\bullet(f,M) \subseteq B_\bullet(f,M)$, $H_\bullet(f,M)$ is annihilated by $B_\bullet(f)$.

(11.13) <u>Corollary</u>: With $I = \operatorname{im} f$ the Koszul homology $H_\bullet(f,M)$ is an A/I-module. In particular, $H_0(f) = A/I$ and $H_0(f,M) = M/IM$.

Define: $K^\bullet(f) = \operatorname{Hom}_A(K_\bullet(f), A)$, $K^\bullet(f,M) = \operatorname{Hom}_A(K_\bullet(f), M)$ and $H^\bullet(f) = H^\bullet(K^\bullet(f))$, $H^\bullet(f,M) = H^\bullet(K^\bullet(f,M))$. $H^\bullet(f)$, $H^\bullet(f,M)$ are called the <u>Koszul cohomology of f</u> (<u>with coefficients in</u> M).

(11.14) <u>Proposition</u>: Let A be a ring, L an A-module, and $f: L \longrightarrow A$ an A-linear map. Set $I = \operatorname{im} f$.

(a) For all $a \in I$ multiplication by a on $K_\bullet(f)$, $K_\bullet(f,M)$, $K^\bullet(f)$, $K^\bullet(f,M)$ is null-homotopic.

(b) I annihilates $H_\bullet(f)$, $H_\bullet(f,M)$, $H^\bullet(f)$, $H^\bullet(f,M)$.

(c) If $I = R$, the complexes $K_\bullet(f)$, $K_\bullet(f,M)$, $K^\bullet(f)$, $K^\bullet(f,M)$ are null-homotopic. In particular, their (co)homology vanishes.

<u>Proof</u>: (a) Let $x \in L$ with $f(x) = a$ and let ϑ_a denote multiplication by a on $K_\bullet(f)$ and λ_x left multiplication by x on $K_\bullet(f)$. Then $\vartheta_a = d_f \circ \lambda_x + \lambda_x \circ d_f$. Thus multiplication by a is null-homotopic and so are $\vartheta_a \otimes M$ and $\operatorname{Hom}_A(\vartheta_a, M)$, the multiplications by a on $K_\bullet(f,M)$ and $K^\bullet(f,M)$.

(b) is a general fact.

(c) Choose $a = 1$ and apply (a) and (b).

Let L_1 and L_2 be A-modules and $f_1 : L_1 \longrightarrow A$, $f_2 : L_2 \longrightarrow A$ A-linear maps. f_1 and f_2 induce a linear form $f : L_1 \oplus L_2 \longrightarrow A$ by $f(x_1 \oplus x_2) = f_1(x_1) + f_2(x_2)$.

(11.15) <u>Proposition</u>: There is an isomorphism of complexes $K_\bullet(f_1) \otimes_A K_\bullet(f_2) \cong K_\bullet(f)$.

<u>Proof</u>: Since $(\Lambda L_1) \otimes_A (\Lambda L_2) \cong \Lambda L$ by (11.6), $K_\bullet(f_1) \otimes_A K_\bullet(f_2) \cong K_\bullet(f)$ as graded A-algebras. The nth graded component of $K_\bullet(f_1) \otimes K_\bullet(f_2)$ is $\bigoplus_{i=0}^{n} (\Lambda^i L_1) \otimes (\Lambda^{n-i} L_2)$ and the differential $d_{f_1} \otimes d_{f_2}$ is defined by:

$$(*) \quad d_{f_1} \otimes d_{f_2}(x \otimes y) = d_{f_1}(x) \otimes y + (-1)^i x \otimes d_{f_2}(y)$$

for $x \in \Lambda^i L_1$ and $y \in \Lambda^{n-i} L_2$. d_f is an antiderivation on ΛL which coincides with $d_{f_1} \otimes d_{f_2}$ on the degree one component $L = L_1 \oplus L_2$. $(*)$ implies that $d_{f_1} \otimes d_{f_2}$ is also an antiderivation on $\Lambda L \cong (\Lambda L_1) \otimes (\Lambda L_2)$. Since an antiderivation on ΛL is uniquely determined by its values on L, $d_f = d_{f_1} \otimes d_{f_2}$ (provided ΛL and $(\Lambda L_1) \otimes (\Lambda L_2)$ are identified).

(11.16) <u>Proposition</u>: Let A be a ring, L an A-module, and $f : L \longrightarrow A$ an A-linear map. Suppose that $\varphi : A \longrightarrow B$ is a homomorphism of rings.

(a) There is a natural isomorphism $K_\bullet(f) \otimes_A B \cong K_\bullet(f \otimes B)$.

(b) If φ is flat, then $H_\bullet(f, M) \otimes_A B \cong H_\bullet(f \otimes B, M \otimes_A B)$ for all A-modules M.

<u>Proof</u>: (a) By (11.5) $(\Lambda L) \otimes_A B \cong \Lambda(L \otimes_A B)$ and $d_f \otimes B$ and $d_{f \otimes B}$ are antiderivations on $\Lambda(L \otimes_A B)$ which coincide in degree one. Thus $d_f \otimes B = d_{f \otimes B}$ by the same argument as in the proof of (11.15).

(b) If $C_\bullet$ is any complex of A-modules and B is a flat A-algebra, then $H_\bullet(C \otimes B) = H_\bullet(C_\bullet) \otimes B$.

Let L and L′ be A-modules with linear forms $f : L \longrightarrow A$ and $f' : L' \longrightarrow A$. By (11.4) every A-linear map $\varphi : L' \longrightarrow L$ induces a unique homomorphism of A-algebras

$\Lambda\varphi : \Lambda L \longrightarrow \Lambda L'$. If $f = f' \circ \varphi$, then $\Lambda\varphi$ is a homomorphism of Koszul complexes, i.e. for all n the diagram

$$\begin{array}{ccc} \Lambda^n L & \xrightarrow{d_f} & \Lambda^{n-1} L \\ {\scriptstyle \Lambda\varphi}\downarrow & & \downarrow{\scriptstyle \Lambda\varphi} \\ \Lambda^n L' & \xrightarrow{d_{f'}} & \Lambda^{n-1} L' \end{array}$$

commutes.

We just showed:

(11.17) Proposition: With the notation as above, if $f = f' \circ \varphi$ then $\Lambda\varphi$ is a homomorphism of complexes.

§3: THE KOSZUL COMPLEX OF A SEQUENCE

Let L be a finitely generated free A-module with basis $e_1, \ldots, e_n$. A linear form $f: L \longrightarrow A$ is uniquely determined by the values $f(e_i) = x_i$ for $1 \leq i \leq n$. Conversely, given a sequence $\underline{x} = x_1, \ldots, x_n \in A$ there is a linear form $f: L \longrightarrow A$ with $f(e_i) = x_i$ for all $1 \leq i \leq n$. We set $K_\bullet(\underline{x}) = K_\bullet(f)$, $H_\bullet(\underline{x}) = H_\bullet(f)$, $K_\bullet(\underline{x}, M) = K_\bullet(f, M)$, etc if $f: L \longrightarrow A$ is defined by $f(e_i) = x_i$.

Since $L = \bigoplus_{i=1}^{n} Ae_i$ and $f = \oplus f_i$ where $f_i: Ae_i \longrightarrow A$ with $f_i(e_i) = x_i$, by (11.15) $K_\bullet(\underline{x}) = K_\bullet(\underline{x}') \otimes K_\bullet(x_n) = K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$ where $\underline{x}' = x_1, \ldots, x_{n-1}$. Moreover, $K_\bullet(\underline{x})$ is essentially invariant under a permutation of $\underline{x}$.

Set $I = (\underline{x})$ and let $F_\bullet$ be a free resolution of A/I. Since $H_0(\underline{x}) = A/I$, by (7.12) there is a morphism of complexes $\varphi_\bullet: K_\bullet(\underline{x}) \longrightarrow F_\bullet$ with $H_0(\varphi_\bullet) = \mathrm{id}_{A/I}$. $\varphi_\bullet$ is unique up to homotopy.

(11.18) Proposition: Let A be a ring, $\underline{x} = x_1, \ldots, x_n$ a sequence in A, and $I = (\underline{x})$. For all i there are natural A-linear maps: $H_i(\underline{x}, M) \longrightarrow \mathrm{Tor}_i^A(A/I, M)$ and $\mathrm{Ext}_A^i(A/I, M) \longrightarrow H^i(\underline{x}, M)$.

Proof: The morphism of complexes $\varphi_\bullet: K_\bullet(\underline{x}) \longrightarrow F_\bullet$ yields morphisms of complexes $\varphi_\bullet \otimes M: K_\bullet(\underline{x}, M) \longrightarrow F_\bullet \otimes M$ and $\mathrm{Hom}_A(\varphi_\bullet, M): \mathrm{Hom}_A(F_\bullet, M) \longrightarrow K^\bullet(\underline{x}, M)$. Apply (7.4).

(11.19) Remark: Let $\underline{x} = x_1, \ldots, x_n \subseteq A$, $I = (\underline{x})$, and M an A-module. Then $H_0(\underline{x}, M) \cong M/IM$ and $H_n(\underline{x}, M) \cong 0 :_M I$.

Proof: For the second claim notice that $K_\bullet(\underline{x}, M): 0 \longrightarrow M \xrightarrow{d_n} M^n \longrightarrow \cdots$ with $d_n(m) = (x_1 m, -x_2 m, \ldots, (-1)^{n+1} x_n m)$. Thus $H_n(\underline{x}, M) = \ker d_n = \{ m \in M \mid x_i m = 0 \text{ for all } 1 \leq i \leq n \}$.

Let M be an A-module. Then $M^* = \mathrm{Hom}_A(M,A)$ denotes the <u>dual</u> of M. Let L be a free A-module with basis $e_1, —, e_n$. Then $e_1 \wedge — \wedge e_n$ is a basis of $\Lambda^n L$. Thus there is an A-isomorphism $\omega_n : \Lambda^n L \xrightarrow{\sim} A$ with $\omega_n(e_1 \wedge — \wedge e_n) = 1$. ($\omega_n$ is an <u>orientation</u> of L) For all $0 \le i \le n$ the bilinear forms:
$\Lambda^i L \times \Lambda^{n-i} L \xrightarrow{\wedge} \Lambda^n L \xrightarrow{\omega_n} A$ induce A-linear map $\omega_i : \Lambda^i L \longrightarrow (\Lambda^{n-i} L)^*$ defined by $(\omega_i(x))(y) = \omega_n(x \wedge y)$ for all $x \in \Lambda^i L$, $y \in \Lambda^{n-i} L$. For $I \subseteq \{1, —, n\}$ with $|I| = i$ write $I' = \{1, —, n\} - I$ and let $J \subseteq \{1, —, n\}$ with $|J| = n-i$. Then

$$e_I \wedge e_J = \mathrm{sgn}(I,J)\, e_{I \cup J} = \begin{cases} 0 & \text{if } J \neq I' \\ \mathrm{sgn}(I,I')\, e_{\{1,—,n\}} & \text{if } J = I'. \end{cases}$$

This implies that for all $0 \le i \le n$ $\omega_i : \Lambda^i L \longrightarrow (\Lambda^{n-i} L)^*$ is an isomorphism. Set $u_i = (-1)^{\binom{i}{2}} \omega_i$ and $u_\bullet = \oplus u_i : \Lambda L \longrightarrow (\Lambda L)^*$.

(11.20) <u>Proposition</u>: (Self-duality) Let $\underline{x} = x_1, —, x_n \subseteq A$ and M an A-module.
(a) $u_\bullet : K_\bullet(\underline{x}) \longrightarrow (K^\bullet(\underline{x}))(-n)$ is an isomorphism of complexes.
(b) $K_\bullet(\underline{x}, M) \cong (K^\bullet(\underline{x}, M))(-n)$
(c) $H_i(\underline{x}, M) \cong H^{n-i}(\underline{x}, M)$

<u>Proof</u>: (a) We need to show that $u_\bullet$ is a morphism of complexes. This follows once we have shown that $\omega_{i-1} \circ d_i = (-1)^{i-1} d^*_{n-i+1} \circ \omega_i$ where $d_\bullet = d_{\underline{x}}$. The latter follows from the identity $\mathrm{sgn}(\nu, I - \{\nu\}) \cdot \mathrm{sgn}(I - \{\nu\}, I' - \{\nu\}) = (-1)^{i-1} \mathrm{sgn}(\nu, I')\, \mathrm{sgn}(I, I')$ where $\nu \in I \subseteq \{1, —, n\}$ and $|I| = i$.
(b) follows from (a) and (c) follows from (b).

(11.21) <u>Proposition</u>: Let $\underline{x} = x_1, —, x_n \subseteq A$ and $0 \to M' \to M \to M'' \to 0$ an exact sequence of A-modules. The induced sequence $0 \to K_\bullet(\underline{x}, M') \to K_\bullet(\underline{x}, M) \to K_\bullet(\underline{x}, M'') \to 0$ is an exact sequence of complexes. This induces a longexact sequence of homology:

$$\cdots \to H_i(\underline{x}, M') \to H_i(\underline{x}, M) \to H_i(\underline{x}, M'') \to H_{i-1}(\underline{x}, M') \to \cdots$$

Proof: Exactness follows from the fact that $\Lambda^i A^n$ are free A-modules. For the longexact sequence see (7.6).

(11.22) Proposition: Let $x \in A$ and $C_\bullet$ a complex of A-modules.

(a) There is an exact sequence of morphisms of complexes:

$$0 \longrightarrow C_\bullet \longrightarrow C_\bullet \otimes K_\bullet(x) \longrightarrow C_\bullet(-1) \longrightarrow 0$$

(b) The induced longexact sequence of the homology is

$$\cdots \longrightarrow H_i(C_\bullet) \longrightarrow H_i(C_\bullet \otimes K_\bullet(x)) \longrightarrow H_{i-1}(C_\bullet) \xrightarrow{(-1)^{i-1}x} H_{i-1}(C_\bullet) \longrightarrow \cdots$$

(c) If x is a NZD on $C_\bullet$, $H_\bullet(C_\bullet \otimes K_\bullet(x)) \cong H_\bullet(C_\bullet \otimes A/(x))$.

Proof: (a) The commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "id"] \arrow[d, "x"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "id"] & A \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

yields an exact sequence of complexes $0 \longrightarrow A \longrightarrow K_\bullet(x) \longrightarrow A(-1) \longrightarrow 0$. Tensor with $C_\bullet \otimes_A$

(b) We only have to show that the connecting homomorphism is multiplication by $\pm x$. Let $\partial_\bullet$ be the differential on $C_\bullet$. In degrees i and $i-1$, the exact sequence of (a) looks like

$$\begin{tikzcd}
0 \arrow[r] & C_i \arrow[r, "nat"] \arrow[d, "\partial_i"'] & C_i \oplus C_{i-1} \arrow[r, "nat"] \arrow[d, "\varphi_i"] & C_{i-1} \arrow[r] \arrow[d, "\partial_{i-1}"] & 0 \\
0 \arrow[r] & C_{i-1} \arrow[r, "nat"] & C_{i-1} \oplus C_{i-2} \arrow[r, "nat"] & C_{i-2} \arrow[r] & 0
\end{tikzcd}$$

where φ_i is given by the matrix $\begin{bmatrix} \partial_i & (-1)^{i-1}x \\ 0 & \partial_{i-1} \end{bmatrix}$.

The claim follows from the construction of the connecting homomorphism (7.6).

(c) Since x is a NZD, $0 \longrightarrow C_\bullet \xrightarrow{x} C_\bullet \longrightarrow C_\bullet \otimes A/(x)$ is exact. On the other hand, the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & K_\bullet(x) \arrow[d] \\
A \arrow[r] & H_0(x) = A/(x)
\end{tikzcd}$$

induces a commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & C_\cdot \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & C_\cdot \otimes K_\cdot(x) \arrow[r] \arrow[d] & C_\cdot(-1) \arrow[r] & 0 \\
0 \arrow[r] & C_\cdot \arrow[r, "x"] & C_\cdot \arrow[r] & C_\cdot \otimes A/(x) & &
\end{tikzcd}$$

, which gives

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H_i(C_\cdot) \arrow[r, "(-1)^{i+1}x"] \arrow[d, "(-1)^{i+1}", "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & H_i(C_\cdot) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & H_i(C_\cdot \otimes K_\cdot(x)) \arrow[r] \arrow[d, "\psi_i"] & H_{i-1}(C_\cdot) \arrow[r, "(-1)^i x"] \arrow[d, "(-1)^i", "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{i-1}(C_\cdot) \arrow[r] \arrow[d, equal] & \cdots \\
\arrow[r] & H_i(C_\cdot) \arrow[r, "x"] & H_i(C_\cdot) \arrow[r] & H_i(C_\cdot \otimes A/(x)) \arrow[r] & H_{i-1}(C_\cdot) \arrow[r, "x"] & H_{i-1}(C_\cdot) \arrow[r] & \cdots
\end{tikzcd}$$

It is easy to check that the middle diagram commutes. Thus ψ_i is an isomorphism by the Five Lemma.

(11.23) Corollary: Let $\underline{x} = x_1, \ldots, x_n \in A$ and M an A-module.

(a) Let $\underline{x}' = x_1, \ldots, x_{n-1}$. There is an exact sequence:

$$\cdots \longrightarrow H_i(\underline{x}', M) \longrightarrow H_i(\underline{x}, M) \longrightarrow H_{i-1}(\underline{x}', M) \xrightarrow{\pm x_n} H_{i-1}(\underline{x}', M) \longrightarrow \cdots$$

(b) Let $\underline{x}' = x_1, \ldots, x_s$, $\underline{x}'' = x_{s+1}, \ldots, x_n$ and assume that $\underline{x}'$ is weakly M-regular. Then $H_\cdot(\underline{x}, M) \cong H_\cdot(\underline{x}'', M \otimes A/(\underline{x}'))$.

Proof: (a) $K_\cdot(\underline{x}, M) = K_\cdot(\underline{x}) \otimes M \cong K_\cdot(\underline{x}') \otimes K_\cdot(x_n) \otimes M \cong K_\cdot(\underline{x}', M) \otimes K_\cdot(x_n)$. Apply (11.22).

(b) By induction on s we may assume that $s=1$. We may permute x_1 in the sequence $\underline{x}$ and $K_\cdot(\underline{x}, M) = K_\cdot(\underline{x}'', x_1, M) \cong K_\cdot(\underline{x}'', M) \otimes K_\cdot(x_1)$. Thus $H_\cdot(\underline{x}, M) = H_\cdot(K_\cdot(\underline{x}'', M) \otimes K_\cdot(x_1)) \cong H_\cdot(K_\cdot(\underline{x}'', M) \otimes A/(x_1))$ by (11.22)(c), since x_1 is a NZD on $K_\cdot(\underline{x}'', M) \cong \bigoplus M$. But $H_\cdot(K_\cdot(\underline{x}'', M) \otimes A/(x_1)) \cong H_\cdot(\underline{x}'', M \otimes_A A/(x_1))$.

(11.24) Corollary: Let $\underline{x} = x_1, \ldots, x_n \in A$ and M an A-module.

(a) If $\underline{x}$ is weakly M-regular, then $K_\cdot(\underline{x}, M)$ is acyclic.

(b) If $\underline{x}$ is A-regular, then $K_\cdot(\underline{x})$ is a free A-resolution of $A/(\underline{x})$.

Proof: Use (11.23)(b).

(11.25) <u>Proposition</u>: Let $\underline{x} = x_1, \ldots, x_n \in A$, $I = (\underline{x})$, $\underline{y} = y_1, \ldots, y_m \in I$, and M an A-module. If $\underline{y}$ is M-regular, then $H_i(\underline{x}, M) = 0$ for $i > n-m$ and $H_{n-m}(\underline{x}, M) \cong \operatorname{Hom}_A(A/I, M/(\underline{y})M) \cong \operatorname{Ext}_A^m(A/I, M)$.

<u>Proof</u>: The last isomorphism follows by (10.14), the rest will be shown by induction on m.
If $m = 0$ then $H_i(\underline{x}, M) = 0$ for $i > n$ and by (11.19) $H_n(\underline{x}, M) \cong 0 :_M I \cong \operatorname{Hom}_A(A/I, M)$.
Let $m > 0$ and write $\overline{M} = M/y_1 M$. The exact sequence $0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow \overline{M} \longrightarrow 0$ induces by (11.21) an exact sequence $0 \longrightarrow K_\bullet(\underline{x}, M) \xrightarrow{y_1} K_\bullet(\underline{x}, M) \longrightarrow K_\bullet(\underline{x}, \overline{M}) \longrightarrow 0$ and thus a longexact sequence of homology:

$$H_{i+1}(\underline{x}, M) \xrightarrow{y_1} H_{i+1}(\underline{x}, M) \longrightarrow H_{i+1}(\underline{x}, \overline{M}) \longrightarrow H_i(\underline{x}, M) \xrightarrow{y_1} H_i(\underline{x}, M)$$

Since $y_1 \in I$ annihilates $H_\bullet(\underline{x}, M)$ by (11.14), this longexact sequence breaks up into short exact sequences $0 \longrightarrow H_{i+1}(\underline{x}, M) \longrightarrow H_{i+1}(\underline{x}, \overline{M}) \longrightarrow H_i(\underline{x}, M) \longrightarrow 0$. If $i > n-m$ then $i+1 > n-(m-1)$, hence $H_{i+1}(\underline{x}, \overline{M}) = 0$ by induction hypothesis, and therefore $H_i(\underline{x}, M) = 0$.
Since $H_{n-m+1}(\underline{x}, M) = 0$, $H_{n-m+1}(\underline{x}, \overline{M}) \cong H_{n-m}(\underline{x}, M)$ and by induction hypothesis $H_{n-m+1}(\underline{x}, \overline{M}) \cong \operatorname{Hom}_A(A/I, \overline{M}/(y_2, \ldots, y_m)\overline{M}) \cong \operatorname{Hom}_A(A/I, M/(\underline{y})M)$.

(11.26) <u>Theorem</u>: Let A be a Noetherian ring, M a finitely generated A-module, $\underline{x} = x_1, \ldots, x_n \in A$, $I = (\underline{x})$, and $g = \operatorname{grade}(I, M) = \operatorname{depth}_I M$.
(a) $K_\bullet(\underline{x}, M)$ is exact if and only if $M = IM$.
(b) If $K_\bullet(\underline{x}, M)$ is not exact, then $\max\{i \mid H_i(\underline{x}, M) \neq 0\} = n - g$.

<u>Proof</u>: (a) "$\Rightarrow$": clear since $M/IM = H_0(\underline{x}, M)$. "$\Leftarrow$" By (11.14) $\operatorname{Supp}_A(H_\bullet(\underline{x}, M)) \subseteq V(I) \cap \operatorname{Supp}(M) \overset{(*)}{=} \operatorname{Supp}(M/IM) = \emptyset$, where $(*)$ follows by Nakayama's Lemma.
(b) If $K_\bullet(\underline{x}, M)$ is not exact, by (a) $IM \neq M$ and $g < \infty$. By (11.25) $H_i(\underline{x}, M) = 0$ for $i > n-g$ and $H_{n-g}(\underline{x}, M) \cong \operatorname{Ext}_A^g(A/I, M)$ and $\operatorname{Ext}_A^g(A/I, M) \neq 0$ by (8.16).

(11.27) <u>Lemma</u>: Let $(A, \mathfrak{m})$ be a local Noetherian ring, M a finitely generated A-module, and $\underline{x} = x_1, \ldots, x_n \in \mathfrak{m}$. If $H_s(\underline{x}, M) = 0$, then $H_i(x_1, \ldots, x_j, M) = 0$ for all $i \geq s$, $j \leq n$.

Proof: By induction on n. The case $n=0$ is trivial. For $n>0$ write $\underline{x}' = x_1, \ldots, x_{n-1}$ and consider the longexact sequence of homology from (11.23)(a):

$$\cdots \longrightarrow H_i(\underline{x}', M) \longrightarrow H_i(\underline{x}, M) \longrightarrow H_{i-1}(\underline{x}', M) \to \cdots \longrightarrow H_s(\underline{x}', M) \xrightarrow{\pm x_n} H_s(\underline{x}', M) \longrightarrow H_s(\underline{x}, M) \to \cdots$$

If $H_s(\underline{x}, M) = 0$ then $H_s(\underline{x}', M) = x_n H_s(\underline{x}', M)$ and $H_s(\underline{x}', M) = 0$ by Nakayama's Lemma since $x_n \in m$. By induction hypothesis, $H_i(x_1, \ldots, x_j, M) = 0$ for $i \geq s$, $j \leq n-1$. Since $H_i(\underline{x}', M) = 0$ for $i \geq s$, the above longexact sequence yields $H_i(\underline{x}, M) = 0$ for $i > s$.

(11.28) Corollary: (Rigidity) Let A be a Noetherian ring, M a finitely generated A-module, and $\underline{x} = x_1, \ldots, x_n \in A$. If $H_s(\underline{x}, M) = 0$ then $H_i(\underline{x}, M) = 0$ for all $i \geq s$.

Proof: Let $I = (\underline{x})$. Since $\mathrm{Supp}_A(H_\cdot(\underline{x}, M)) \subseteq V(I)$, we may localize at $p \in V(I)$ to assume that (A, m) is local with $\underline{x} \subseteq m$. Use (11.27).

(11.29) Corollary: Let A be a Noetherian ring, M a finitely generated A-module, $\underline{x} = x_1, \ldots, x_n \in A$, $I = (\underline{x})$, and assume that $IM \neq M$. The following are equivalent:

(a) $\underline{x}$ is M-quasi regular

(b) $K_\cdot(\underline{x}, M)$ is acyclic

(c) $H_1(\underline{x}, M) = 0$

(d) $\mathrm{grade}(I, M) = n$.

Proof: By (8.10) (and homework) (a) $\Leftrightarrow$ (a'): $\underline{x}$ is M_p-regular for all $P \in \mathrm{Supp}_A(M/IM)$.
By (11.16) (b) $\Leftrightarrow$ (b'): $K(\underline{x}, M_p)$ is acyclic for all $P \in \mathrm{Supp}_A(M/IM)$.
(a') $\Rightarrow$ (b'): by (11.24)(a)
(b') $\Rightarrow$ (a'): We may assume that (A, m) is local with $\underline{x} \subset m$. By (11.27) $H_1(x_1, \ldots, x_j, M) = 0$ for $j \leq n$. Thus for all $1 \leq j \leq n$ the sequence

$$H_1(x_1, \ldots, x_j, M) = 0 \longrightarrow H_0(x_1, \ldots, x_{j-1}, M) = M/(x_1, \ldots, x_{j-1})M \xrightarrow{\pm x_j} H_0(x_1, \ldots, x_{j-1}, M) = M/(x_1, \ldots, x_{j-1})M$$

is exact by (11.23). Thus $\underline{x}$ is M-regular
(b) $\Leftrightarrow$ (c): (11.28) and (b) $\Leftrightarrow$ (d) (11.26)(b).

(11.30) <u>Theorem</u>: Let A be a local Noetherian ring and $M \neq 0$ a finitely generated A-module of finite injective dimension. Then $\dim M \leq \operatorname{injdim} M = \operatorname{depth} A$.

<u>Proof</u>: $\dim M \leq \operatorname{injdim} M$. Let $d = \dim M$ and let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ be a chain of primes in $\operatorname{Supp}(M)$. We want to show by induction on i that $\mu_i(P_i, M) \neq 0$. Then $\mu_d(P_d, M) \neq 0$ and $\operatorname{injdim} M \geq d$. For $i = 0$, $P_0 A_{P_0}$ is minimal in $\operatorname{Supp}(M_{P_0})$. Thus $P_0 A_{P_0}$ is an associated prime of M_{P_0}, hence $\operatorname{Hom}_{A_{P_0}}(k(P_0), M_{P_0}) \neq 0$ and $\mu_0(P_0, M) \neq 0$. Suppose $0 \leq i < d$ and $\mu_i(P_i, M) \neq 0$. If $\mu_{i+1}(P_{i+1}, M) = 0$, then $\operatorname{Ext}^{i+1}_{A_{P_{i+1}}}(k(P_{i+1}), M_{P_{i+1}}) = 0$ and by (10.11) $\operatorname{Ext}^{i}_{A_{P_{i+1}}}(A_{P_{i+1}}/P_i A_{P_{i+1}}, M_{P_{i+1}}) = 0$ since $\dim A_{P_{i+1}}/P_i A_{P_{i+1}} = 1$. Localizing at $P_i A_{P_{i+1}}$ yields $\mu_i(P_i, M) = 0$, a contradiction.

$\operatorname{injdim} M = \operatorname{depth} A$: Let $t = \operatorname{depth} A$ and let $\underline{x} = x_1, \dots, x_t$ be an A-regular sequence. By (11.24) $K_\bullet(\underline{x})$ is a free A-resolution of $N = A/(\underline{x})$ of length t. Thus $\operatorname{Ext}^t_A(N, M) = H^t(\operatorname{Hom}_A(K_\bullet(\underline{x}), M)) = H^t(\underline{x}, M) \cong H_0(\underline{x}, M) \cong M/(\underline{x})M \neq 0$ by (11.20) and (11.19). Hence $t = \sup\{n \mid \operatorname{Ext}^n_A(N, M) \neq 0\}$. Since $\operatorname{depth} N = 0$, by (10.13) $t = \operatorname{injdim} M$.