CHAPTER IV: DIMENSION THEORY

§1: GRADED RINGS AND MODULES

Recall: An abelian semigroup G with identity is a nonempty set G together with an associative and commutative operation '+' and a zero element $0 \in G$ such that $g+0 = 0+g = g$ for all $g \in G$. Main examples: $(\mathbb{Z},+)$ and $(\mathbb{N},+)$ (where $0 \in \mathbb{N}$!).

(4.1) Definition: Let A be a ring and G an abelian semigroup with identity.
(a) A is called a ($\underline{G}$-) graded ring if A has a direct sum decomposition as an additive group: $A = \bigoplus_{i\in G} A_i$, where $A_i \subseteq A$ are subgroups of $(A,+)$, such that for all $i,j \in G$:

$$A_i A_j \subseteq A_{i+j}.$$

(b) Let $A = \bigoplus_{i\in G} A_i$ be a graded ring and M an A-module. M is a graded A-module if $(M,+)$ has a direct sum decomposition $M = \bigoplus_{i\in G} M_i$ such that $A_i M_j \subseteq M_{i+j}$ for all $i,j \in G$.

(4.2) Example: Let K be a field. The polynomial ring $A = K[x_1, \ldots, x_r]$ is an $\mathbb{N}$-graded ring, $A = \bigoplus_{n\geq 0} A_n$ where $A_n = \{ \sum_{|(i)|=n} a_{(i)} x_1^{i_1} \cdots x_r^{i_r} \mid a_{(i)} \in K \}$ is the set of homogeneous polynomials of degree n.

(4.3) Definition: Let A be a graded ring and M a graded A-module. An element $a \in A$ ($m \in M$, resp) is called homogeneous if $a \in A_i$ ($m \in M_i$, resp.) for some $i \in G$. In this case i is called the degree of a (m, resp.).

(4.4) Remark: Every element $a \in A = \bigoplus_{i\in G} A_i$ ($m \in \bigoplus_{i\in G} M_i = M$, resp.) can be written (uniquely, up to order) as a sum of homogeneous elements $a = \sum a_i$, $a_i \in A_i$ ($m = \sum_i m_i$, $m_i \in M$, resp.). The a_i (m_i, resp.) are called the homogeneous components of a (m, resp).

(4.5) Definition: Let $A = \bigoplus_{i\in G} A_i$ be a graded ring and $M = \bigoplus_{i\in G} M_i$ a graded A-module. A submodule $N \subseteq M$ is called homogeneous if $N = \bigoplus_{i\in G}(N\cap M_i)$. An ideal I of A is homogeneous if I is a homogeneous submodule of the graded A-module A, that is, if $I = \bigoplus_{i\in G}(I\cap A_i)$.

(4.6) Proposition: Let $A = \bigoplus_{i\in G} A_i$ be a graded ring, $M = \bigoplus_{i\in G} M_i$ a graded A-module and $N \subseteq M$ a submodule. The following are equivalent:

(a) N is a homogeneous submodule of M.

(b) N is generated by homogeneous elements.

(c) For all $m = \sum'_{i\in G} m_i \in M$ with $m_i \in M_i$: $m \in N \iff m_i \in N$ for all $i \in G$.

Proof: (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a): trivial

(b) $\Rightarrow$ (c): Suppose that N is generated by homogeneous elements $\{n_j\}_{j\in J}$ where $n_j \in M_{d_j}$, i.e. $\deg(n_j) = d_j$. We have to show that $N \subseteq \bigoplus_{i\in G}(N\cap M_i)$. Let $n\in N$, then $n = \sum' a_j n_j$ for some $a_j \in A$. For all j: $a_j = \sum' a_{jk}$ with $a_{jk}\in A_k$. Hence $n = \sum'_{j,k} a_{jk} n_j$. The elements $a_{jk}n_j$ are homogeneous. Collect terms of the same degree:

$$m_i = \sum_{\deg(a_{jk}n_j)=i} a_{jk}n_j .$$

The $m_i \in N$ are the homogeneous components of N.

(4.7) Remark: Let $A = \bigoplus_{i\in G} A_i$ be a graded ring, $M = \bigoplus_{i\in G} M_i$ a graded A-module and $N\subseteq M$ a homogeneous submodule. $M/N \cong \bigoplus_{i\in G} M_i/(N\cap M_i)$ is a graded A-module.

From now on: $G = \mathbb{N} = \mathbb{N}_0$.

(4.8) Definition: Let A be a ring. A filtration of A is a chain of ideals:
$\mathcal{F}: A = I_0 \supseteq I_1 \supseteq \dots \supseteq I_n \supseteq \dots$ which satisfies $I_n I_m \subseteq I_{n+m}$ for all $n,m\in\mathbb{N}$.

(4.9) Remark and Definition: Let $\mathcal{F}: A = I_0 \supseteq I_1 \supseteq \dots \supseteq I_n \supseteq \dots$ be a filtration of the ring A.

Consider the (abelian) additive group: $gr_{\mathcal{F}}(A) = \bigoplus_{n\geq 0} I_n/I_{n+1}$. Define a multiplication on $gr_{\mathcal{F}}(A)$ as follows: Let $\alpha \in I_n/I_{n+1}$ and $\beta \in I_m/I_{m+1}$ with representatives $a \in I_n$ and $b \in I_m$. Define $\alpha\beta = ab + I_{n+m+1} \in I_{n+m}/I_{n+m+1}$. This multiplication is well defined and extends linearly to a multiplication on $gr_{\mathcal{F}}(A)$. $gr_{\mathcal{F}}(A)$ is a commutative graded ring. $gr_{\mathcal{F}}(A)$ is called the <u>graded</u> <u>ring</u> <u>associated</u> <u>to</u> <u>the</u> <u>filtration</u> $\mathcal{F}$.
If $I \subseteq A$ is an ideal the filtration $\mathcal{F}$: $A = I^0 \supseteq I^1 \supseteq I^2 \supseteq \dots \supseteq I^n \supseteq \dots$ is called the I-<u>adic</u> <u>filtration</u> of A. $gr_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is called the <u>associated</u> <u>graded</u> <u>ring</u>.

(4.10) <u>Remark</u>: Let $A = \bigoplus_{n\geq 0} A_n$ be a graded ring. $A_0 \subseteq A$ is a subring of A and $A_+ = \bigoplus_{n\geq 1} A_n$ is an ideal of A with $A/A_+ \cong A_0$.

(4.11) <u>Example</u>: Let A be a ring and $I = (a_1, \dots, a_n) \subseteq A$ a finitely generated ideal. Then $gr_I(A)$ is a homomorphic image of the polynomial ring $(A/I)[x_1, \dots, x_n]$. In particular, $gr_I(A)$ is a Noetherian ring if A is Noetherian.

<u>Proof</u>: There is a surjective homomorphism of rings $\varphi: (A/I)[x_1, \dots, x_n] \longrightarrow \bigoplus_{n\geq 0} I^n/I^{n+1}$ defined by $\varphi|_{A/I} = \mathrm{id}_{A/I}$ and $\varphi(x_i) = a_i + I^2 \in I/I^2$.

(4.12) <u>Proposition</u>: Let $A = \bigoplus_{n\geq 0} A_n$ be a graded ring. The following are equivalent:
(a) A is Noetherian.
(b) A_0 is Noetherian and A is a finitely generated A_0-algebra.

<u>Proof</u>: (b) $\Rightarrow$ (a): Hilbert's Basis Theorem
(a) $\Rightarrow$ (b): Since $A_0 \cong A/A_+$, A_0 is Noetherian. $A_+ = \bigoplus_{n\geq 1} A_n$ is a finitely generated ideal of A. Assume $A_+ = (a_1, \dots, a_r)$ where $a_i \in A_{d_i}$ homogeneous.
<u>Claim:</u> $A = A_0[a_1, \dots, a_r]$.
<u>Pf of Cl</u>: We want to show by induction on n that $A_n \subseteq A_0[a_1, \dots, a_r]$.
Let $b \in A_n$. Then $b = \sum_{i=1}^{r} f_i a_i$ for some $f_i \in A$. Let g_i be the homogeneous

component of degree $n-d_i$ of f_i if $n \geq d_i$ and $g_i = 0$ if $n < d_i$. Then $b = \sum_{i=1}^{r} g_i a_i$ and by induction hypothesis $g_i \in A_0[a_1, \ldots, a_r]$. Thus $b \in A_0[a_1, \ldots, a_r]$.

(4.13) <u>Definition</u>: Let A be a ring, $I \subseteq A$ an ideal, M an A-module and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq \ldots$ a descending chain of submodules of M.

(a) $\mathcal{F}$ is called a <u>filtration</u> of M.

(b) $\mathcal{F}$ is called an I-<u>filtration</u> of M if $IM_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$.

(c) $\mathcal{F}$ is called a <u>stable</u> I-<u>filtration</u> of M if $\mathcal{F}$ is an I-filtration and if there is an $n_0 \in \mathbb{N}$ so that $IM_n = M_{n+1}$ for all $n \geq n_0$.

(4.14) <u>Remark</u> and <u>Definition</u>: Let M be an A-module and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq \ldots$ an I-filtration of M. The <u>graded module associated</u> to $\mathcal{F}$: $gr_{\mathcal{F}}(M) = \bigoplus_{n \geq 0} M_n/M_{n+1}$ is naturally a graded $gr_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ - module. The I-filtration: $M \supseteq IM \supseteq I^2M \supseteq \ldots$ defined by the powers of I is called the I-<u>adic filtration</u> of M. $gr_I(M) = \bigoplus_{n \geq 0} I^nM/I^{n+1}M$ is called the <u>associated graded module</u>. $gr_I(M)$ is a graded $gr_I(R)$-module.

(4.15) <u>Definition</u>: Let A be a ring and $I \subseteq A$ an ideal. The graded ring $\mathcal{R} = \bigoplus_{n \geq 0} I^n$ is called the <u>Rees algebra</u> associated to I.

(4.16) <u>Remark</u>: Let A be a ring, $I \subseteq A$ an ideal, M an A-module, and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ an I-filtration of M. Then $\mathcal{M} = \bigoplus_{n \geq 0} M_n$ is a graded module over the Rees algebra $\mathcal{R} = \bigoplus_{n \geq 0} I^n$.

(4.17) <u>Proposition</u>: Let A be a ring, Noetherian, $I \subseteq A$ an ideal, M a finitely generated A-module, and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ an I-filtration of M. The following are equivalent:

(a) $\mathcal{F}$ is a stable I-filtration

(b) $\mathcal{M} = \bigoplus_{n \geq 0} M_n$ is a finitely generated $\mathcal{R} = \bigoplus_{n \geq 0} I^n$-module.

<u>Proof</u>: (a) $\Rightarrow$ (b): Let $n_0 \in \mathbb{N}$ with $IM_n = M_{n+1}$ for all $n \geq n_0$. $\mathcal{M}$ is generated over $\mathcal{R}$ by

$M_0 \oplus M_1 \oplus \ldots \oplus M_{n_0}$. The statement follows since every M_i is a finitely generated A-module.
(b) $\Rightarrow$ (a): Suppose that $\mathcal{M} = \mathcal{R} m_1 + \ldots + \mathcal{R} m_r$ where $m_i \in M_{n_i}$. For $n > \max\{n_1, \ldots, n_r\}$:
$M_n = (\mathcal{M})_n = \{ \sum_{i=1}^{r} a_i m_i \mid a_i \in \mathcal{R}_{n-n_i} \} = \{ \sum_{i=1}^{r} a_i m_i \mid a_i \in I^{n-n_i} \} \subseteq I M_{n-1}$.
Thus $M_n \subseteq I M_{n-1} \subseteq M_n$ and $I M_{n-1} = M_n$ for all $n > n_0 = \max\{n_1, \ldots, n_r\}$.

(4.18) Theorem: (Artin–Rees) Let A be a Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A-module, and $N \subseteq M$ a submodule. The I-filtration $\{I^n M \cap N\}_{n \geq 0}$ of N is I-stable, that is, there is an integer $n_0 \in \mathbb{N}$ such that $I^n M \cap N = I^{n-n_0}(I^{n_0} M \cap N)$ for all $n \geq n_0$.

Proof: Obviously, $\{I^n M \cap N\}_{n \geq 0}$ is an I-filtration and $\mathcal{N} = \bigoplus_{n \geq 0} (I^n M \cap N)$ is a homogeneous submodule of the $\mathcal{R} = \bigoplus_{n \geq 0} I^n$-module $\mathcal{M} = \bigoplus_{n \geq 0} I^n M$. Since $\{I^n M\}_{n \geq 0}$ is a stable I-filtration of M, $\mathcal{M}$ is a finitely generated $\mathcal{R}$-module by (4.17). It remains to show that $\mathcal{R}$ is Noetherian. Then $\mathcal{N}$ is a finitely generated $\mathcal{R}$-module and by (4.17) $\{I^n M \cap N\}$ is a stable I-filtration of N.

(4.19) Lemma: Let A be a Noetherian ring and $I \subseteq A$ a (proper) ideal. The Rees-algebra $\mathcal{R} = \bigoplus_{n \geq 0} I^n$ is Noetherian.

Proof: Suppose that $I = (a_1, \ldots, a_s)$. $\mathcal{R}$ can be identified with the finitely generated subalgebra $\mathcal{R} \cong A[a_1 x, \ldots, a_s x]$ of the polynomial ring $A[x]$.

(4.20) Corollary: (Krull's intersection theorem) Let A be a Noetherian ring, $I \subseteq A$ an ideal, and M a finitely generated A-module. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n N$ for all $n \in \mathbb{N}$.

Proof: By (4.18) there is an $r \in \mathbb{N}$ so that $I^n(I^r M \cap N) = I^{n+r} M \cap N$ for all $n \in \mathbb{N}$.
By construction of N: $N \subseteq I^{r+n} M \cap N$ and thus:

$$N \subseteq I^{r+n} M \cap N = I^n (I^r M \cap N) \subseteq I^n N \subseteq N.$$

(4.21) <u>Corollary</u>: Let A be a Noetherian ring, $I \subseteq \mathrm{Jac}(A)$ an ideal in the Jacobson radical of A, and M a finitely generated A-module. Then: $\bigcap_{n \geq 0} I^n M = (0)$.

<u>Proof</u>: Lemma of Nakayama

(4.22) <u>Corollary</u>: Let A be a Noetherian ring and $I \subseteq \mathrm{Jac}(A)$ an ideal. Then:

$$\bigcap_{n \geq 0} I^n = (0).$$

§2: HILBERT FUNCTIONS

(4.23) Lemma: Let $A = \bigoplus_{n\geq 0} A_n$ be a Noetherian graded ring and $M = \bigoplus_{n\geq 0} M_n$ a finitely generated graded A-module. For all $n \in \mathbb{N}$ M_n is a finitely generated A_0-module.

Proof: 1. case: $M = R$

The proof is by induction on n. By (4.12) A is a finitely generated A_0-algebra. Let $a_1, \ldots, a_r \in A$ be homogeneous elements with $\deg a_i = d_i > 0$ so that $A = A_0[a_1, \ldots, a_r]$. For $n > 0$: $A_n = a_1 A_{n-d_1} + \ldots + a_r A_{n-d_r}$ where $A_i = 0$ if $i < 0$. By induction hypothesis A_j, for $j < n$, is a finitely generated A_0-module, thus A_n is a finitely generated A_0-module.

2. case: M any finitely generated graded A-module.

M is generated by finitely many homogeneous elements: $M = Am_1 + \ldots + Am_s$ where $\deg(m_i) = e_i \geq 0$. Then $M_n = A_{n-e_1} m_1 + \ldots + A_{n-e_s} m_s$ where $A_i = 0$ if $i < s$.
M_n is a finitely generated A_0-module since A_j is finitely generated.

(4.24) Remark and Definition: Let $A = \bigoplus_{n\geq 0} A_n$ be a Noetherian graded ring with A_0 an Artinian ring. Let $M = \bigoplus_{n\geq 0} M_n$ be a finitely generated graded A-module. By (4.23) every M_n is an A_0-module of finite length, i.e. $\ell_{A_0}(M_n) < \infty$.
The Hilbert series $P(M,t)$ of M is defined by:

$$P(M,t) = \sum_{n=0}^{\infty} \ell_{A_0}(M_n)\, t^n \in \mathbb{Z}[\![t]\!] \qquad (t \text{ a variable}).$$

(4.25) Lemma: Let A be an Artinian ring and

$$(*) \qquad 0 \longrightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots\ldots \xrightarrow{\varphi_{n-1}} M_n \longrightarrow 0$$

an exact sequence of finitely generated A-modules. Then $\sum_{i=1}^{n} (-1)^i \ell_A(M_i) = 0$.

Proof: By induction on n. (1.79) provides the formula if $n = 3$. For $n > 3$ split $(*)$ into (short) exact sequences: $0 \longrightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow \cdots \xrightarrow{\varphi_{n-3}} M_{n-2} \longrightarrow \operatorname{im}(\varphi_{n-2}) = \ker(\varphi_{n-1}) \longrightarrow 0$

and $\quad 0 \longrightarrow \ker(\varphi_{n-1}) \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow 0$.

By induction hypothesis: $\quad \sum_{i=1}^{n-2} (-1)^i \ell_A(M_i) + (-1)^{n-1} \ell_A(\ker(\varphi_{n-1})) = 0$

and by (1.79): $\quad \ell_A(\ker(\varphi_{n-1})) = \ell_A(M_{n-1}) + (-1)\,\ell_A(M_n)$.

The statement follows.

(4.26) __Theorem__: Let $A = \bigoplus_{n\geq 0} A_n$ be a Noetherian graded ring. Assume that A_0 is Artinian and that $A = A_0[a_1, \ldots, a_r]$ where the a_i are homogeneous with $\deg a_i = d_i > 0$. Let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated graded A-module. The Hilbert series $P(M,t)$ is a rational function of the form: $\quad P(M,t) = \dfrac{f(t)}{\prod_{i=1}^{r} (1-t^{d_i})} \in \mathbb{Q}(t)$

where $f(t) \in \mathbb{Z}[t]$ is a polynomial with integer coefficients.

__Proof__: By induction on r.

$r=0$: Then $A = A_0$ and M is a finitely generated module over the Artinian ring A_0. Thus M is a Noetherian A_0-module and the chain of submodules: $M_0 \subseteq M_0 \oplus M_1 \subseteq M_0 \oplus M_1 \oplus M_2 \subseteq \ldots$ is stationary. This implies $M_n = 0$ for all $n \geq n_0$ and $P(M,t) \in \mathbb{Z}[t]$.

$r>0$: For all $n \in \mathbb{N}$ the map $\varphi_{n,r}: M_n \longrightarrow M_{n+d_r}$ defined by $\varphi_{n,r}(m) = a_r m$ is A_0-linear. This yields for all $n \in \mathbb{N}$ an exact sequence of A_0-modules:

$(*_n)$: $\quad 0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{\varphi_{n,r}} M_{n+d_r} \longrightarrow L_{n+d_r} \longrightarrow 0$

where $K_n = \ker(\varphi_{n,r})$ and $L_{n+d_r} = \operatorname{coker}(\varphi_{n,r}) = M_{n+d_r}/a_r M_n$. Define:

$K = \bigoplus_{n\geq 0} K_n$ and $L = \bigoplus_{n \geq 0} L_{n+d_r}$. Then:

(a) K is a homogeneous submodule of M.

(b) $L = (\bigoplus_{n \geq d_r} M_n)/a_r M$ is a graded A-module.

(c) $A/a_r A = A_0[\bar{a}_1, \ldots, \bar{a}_{r-1}] = \bigoplus_{n\geq 0} A'_n$ is a graded ring which is generated over $A_0 = A'_0$ by $r-1$ homogeneous elements.

(d) Since $a_r K = 0$ and $a_r L = 0$, the A-modules K and L can be considered as finitely generated (A/a_rA)-modules.

The induction hypothesis applies to K and L and there are polynomials $g(t), h(t) \in \mathbb{Z}[t]$ so that:

$$P(K,t) = \sum_{n=0}^{\infty} \ell_{A_0}(K_n)\, t^n = \frac{g(t)}{\prod_{i=1}^{r-1}(1-t^{d_i})} \quad \text{and} \quad P(L,t) = \sum_{n=0}^{\infty} \ell_{A_0}(L_{n+d_r})\, t^n = \frac{h(t)}{\prod_{i=1}^{r-1}(1-t^{d_i})}.$$

By (4.25) we obtain from $(*_n)$: $\ell(K_n) - \ell(M_n) + \ell(M_{n+d_r}) - \ell(L_{n+d_r}) = 0$.

Multiplying by t^{n+d_r}: $\quad t^{d_r}\ell(K_n)t^n - t^{d_r}\ell(M_n) + \ell(M_{n+d_r})t^{n+d_r} - t^{d_r}\ell(L_{n+d_r})t^n = 0$

and taking sums yields: $t^{d_r}P(K,t) - t^{d_r}P(M,t) + P(M,t) - t^{d_r}P(L,t) = k(t)$, where $k(t) = \sum_{i=0}^{d_r-1} \ell(M_i)\, t^i$. This proves the statement.

(4.27) <u>Remark</u>: The function $(1-t)^{-1}$ can be presented as a power series by $(1-t)^{-1} = \sum_{n=0}^{\infty} t^n \in \mathbb{Z}[\![t]\!]$
By taking the d^{th} derivative we obtain: $\quad (1-t)^{-d} = \sum_{n=0}^{\infty} \binom{d+n-1}{d-1} t^n \in \mathbb{Z}[\![t]\!]$.

(4.28) <u>Theorem:</u> Let $A = \bigoplus_{n\geq 0} A_n$ be a Noetherian graded ring with A_0 Artinian. Suppose that $A = A_0[a_1, \ldots, a_r]$ with a_i homogeneous of degree one, that is, A is generated in degree one. Let $M = \bigoplus_{n\geq 0} M_n$ be a finitely generated graded A-module. Then there is a polynomial $\varphi_M \in \mathbb{Q}[t]$ and an integer $n_0 \in \mathbb{N}$ such that $\ell_{A_0}(M_n) = \varphi_M(n)$ for all $n \geq n_0$.

(4.29) <u>Definition</u>: Assumptions as in (4.28). The function $f_M : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f_M(n) = \ell_{A_0}(M_n)$ is called the <u>Hilbert function</u> of M. The polynomial φ_M of (4.28) is called the <u>Hilbert polynomial</u> of M.

<u>Proof</u> of (4.28): By (4.26): $P(M,t) = \dfrac{f(t)}{(1-t)^r} = \dfrac{g(t)}{(1-t)^d}$ where $g(t) \in \mathbb{Z}[t]$ with $g(1) \neq 0$.
Apply the formula: $\quad (1-t)^{-d} = \sum_{n=0}^{\infty} \binom{d+n-1}{d-1} t^n \in \mathbb{Z}[\![t]\!]$ and suppose that
$g(t) = b_s t^s + b_{s-1} t^{s-1} + \ldots + b_0 \in \mathbb{Z}[t]$ where $b_i \in \mathbb{Z}$. Then

$$(*) \qquad \ell_{A_0}(M_n) = b_0\binom{d+n-1}{d-1} + b_1\binom{d+n-2}{d-1} + \ldots + b_s\binom{d+n-s-1}{d-1}$$

where $\binom{m}{d-1} = 0$ if $m < d-1$. For $n \geq s+1-d$ the right hand side of $(*)$ is a polynomial $\varphi_M(n)$ in n. Since $g(1) \neq 0$, $\varphi_M(t)$ is a polynomial of degree $d-1$ with leading coefficient $g(1)/(d-1)!$ and $\varphi_M(n) = \ell_{A_0}(M_n)$ for all $n \geq s+1-d$.

(4.30) Remark: The degree $d-1$ of $\varphi_M(t)$ is strictly less than r, the number of generators of A over A_0.

(4.31) Examples: (a) Let A_0 be an Artinian ring and $A = A_0[x_0, \ldots, x_r]$ the polynomial ring in $r+1$ variables over A_0. Consider the natural grading $A = \bigoplus_{n\geq 0} A_n$ where $A_n = \{ \sum_i a_{(i)} x_0^{i_0} \ldots x_r^{i_r} \mid |i| = n$ and $a_{(i)} \in A_0 \}$. The number of monomials of degree n in $r+1$ variables is $\binom{n+r}{r}$. Therefore:

$\ell_{A_0}(A_n) = \ell_{A_0}(A_0)\binom{n+r}{r}$ and $\varphi_A(t) = \ell_{A_0}(A_0)\binom{t+r}{r} = \left(\ell_{A_0}(A_0)/r!\right)(t+r)(t+r-1)\ldots(t+1)$.

(b) Let K be a field and $f(x_0, \ldots, x_r) = \sum_{|i|=s} a_{(i)} x_0^{i_0} \ldots x_s^{i_s} \in K[x_0, \ldots, x_r]$ a homogeneous polynomial of degree s. Put $R = K[x_0, \ldots, x_r] = \bigoplus_{n\geq 0} R_n$ where $R_n = \{ \sum_{|i|=n} b_{(i)} x_0^{i_0} \ldots x_r^{i_r} \mid b_{(i)} \in K \}$ is the set of homogeneous polynomials of degree n. Let $A = R/(f) = \bigoplus_{n\geq 0} A_n$ where $A_n = R_n$ if $n<s$ and $A_n = R_n/R_{n-s}f$ if $n \geq s$. Hence for $n \geq s$:

$$\ell_K(A_n) = \ell_K(R_n) - \ell_K(R_{n-s}f) = \ell(R_n) - \ell(R_{n-s}) = \binom{n+r}{r} - \binom{n-s+r}{r}.$$

Let $\binom{t+r}{r} = 1/r!\, t^r + c_{r-1}t^{r-1} + \ldots + c_0$ with $c_i \in \mathbb{Q}$ then

$$\varphi_A(t) = 1/r!\,(t^r - (t-s)^r) + c_{r-1}(t^{r-1} - (t-s)^{r-1}) + \ldots + c_0$$
$$= s/(r-1)!\, t^{r-1} + \text{terms of lower degree}.$$

In particular, $\deg(\varphi_A(t)) = r-1 = \dim(A) - 1$.

(4.31) Remark: If $f(t) \in \mathbb{Q}[t]$ is a polynomial with $f(n) > 0$ for all but finitely many $n \in \mathbb{N}$, the leading coefficient of f is greater than 0. In particular, the Hilbert polynomial $\varphi_M(t) \in \mathbb{Q}[t]$ has a positive leading coefficient.

(4.32) Definition: Let A be a Noetherian semilocal ring. An ideal $I \subseteq A$ is called an ideal of definition of A if there is an $m \in \mathbb{N}$ such that $\mathrm{Jrad}(A)^m \subseteq I \subseteq \mathrm{Jrad}(A)$ (or equivalently, if $\mathrm{rad}(I) = \mathrm{Jrad}(A)$.)

(4.33) Lemma: Let A be a Noetherian semilocal ring and $I \subseteq A$ an ideal. The following conditions are equivalent:

(a) I is an ideal of definition of A.

(b) $I \subseteq \mathrm{Jrad}(A)$ and A/I is Artinian

(c) $I \subseteq \mathrm{Jrad}(A)$ and A/I has finite length.

(d) $\mathrm{Supp}_A(A/I) = \text{m-}\mathrm{Spec}(A)$.

Proof: Homework

(4.34) Remark: If A is a local Noetherian ring with maximal ideal m conditions (a)-(d) of (4.33) are equivalent to I is m-primary.

(4.35) Lemma: Let A be a semilocal Noetherian ring, $I \subseteq A$ an ideal of definition, and M a finitely generated A-module. Suppose that $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ is an I-filtration of M. Then $\ell_A(M/M_n) < \infty$ for all $n \in \mathbb{N}$.

Proof: For all $n \in \mathbb{N}$: $I^n M \subseteq M_n$ and M/M_n is a finitely generated A/I^n-module. Since I is an ideal of definition, so is I^n, and A/I^n is an Artinian ring. Thus $\ell_{(A/I^n)}(M/M_n) = \ell_A(M/M_n) < \infty$.

Goal: Let A be a Noetherian semilocal ring, $I \subseteq A$ an ideal of definition, and M a finitely generated A-module. Suppose that $I = (a_1, \dots, a_r)$ and consider the associated graded ring and module: $gr_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ and $gr_I(M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$.

$gr_I(A)$ is a Noetherian graded ring which is as (A/I)-algebra generated in degree one by $a_1 + I^2, \dots, a_r + I^2 \in I/I^2$. Furthermore, $gr_I(M)$ is a finitely generated graded $gr_I(A)$-module. By (4.28) there is a polynomial $\varphi_M^I \in \mathbb{Q}[t]$ so that for all $n \geq n_0$:

$$\varphi_M^I(n) = \ell_A(I^n M / I^{n+1} M) = \ell_{(A/I)}(I^n M / I^{n+1} M).$$

Using the exact sequence: $0 \to I^nM/I^{n+1}M \to M/I^{n+1}M \to M/I^nM \to 0$

induction on n yields that $\quad l_A(M/I^{n+1}M) = \sum_{i=0}^{n} l_A(I^iM/I^{i+1}M)$. $(*)$

Formula $(*)$ implies that there is a polynomial $\chi_M^I \in \mathbb{Q}[t]$ such that for large n:

$$\chi_M^I(n) = l_A(M/I^{n+1}M).$$

In the remainder of this section we show that the degree of φ_M^I and χ_M^I does not depend on the choice of the ideal of definition I. The next section relates the degree of φ_M^I (resp. of χ_M^I) to the dimension of M ($\dim M = \dim(A/\mathrm{ann}(M))$).

(4.36) Lemma: Let $f: \mathbb{N} \to \mathbb{N}$ be a function. The following are equivalent:

(a) There is a polynomial $P(t) \in \mathbb{Q}[t]$ of degree $\leq r+1$ with $f(n) = P(n)$ for all but finitely many $n \in \mathbb{N}$.

(b) There is a polynomial $Q(t) \in \mathbb{Q}[t]$ of degree $\leq r$ with $f(n+1) - f(n) = Q(n)$ for all but finitely many $n \in \mathbb{N}$.

Proof: Consider the following 'binomial' polynomials in $\mathbb{Q}[t]$:

$\binom{t}{0} = 1$ and for $k \in \mathbb{N} - (0)$: $\binom{t}{k} = \frac{1}{k!} t(t-1)\ldots(t-k+1)$.

Obviously, $\binom{t}{k}$ is a polynomial of degree k and the set $\{\binom{t}{k}\}_{k \in \mathbb{N}_0}$ forms a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}[t]$.

(a) $\Rightarrow$ (b): Let $Q(t) = P(t+1) - P(t)$.

(b) $\Rightarrow$ (a): First observe by induction on n that $\sum_{\nu=0}^{n} \binom{d+\nu-1}{d-1} = \binom{d+n}{d}$.

Write $Q(t) = \sum_{k=0}^{r} a_k \binom{t}{k}$ with $a_k \in \mathbb{Q}$ and let $n_0 \in \mathbb{N}$ with $n_0 \geq r$ such that for all $n \geq n_0$: $f(n+1) - f(n) = Q(n)$. Then:

$$\begin{aligned} f(n+1) &= Q(n) + f(n) = Q(n) + Q(n-1) + f(n-1) = \ldots = \\ &= \sum_{\rho=0}^{n-n_0} Q(n-\rho) + f(n_0) = \sum_{k=0}^{r} a_k \Big(\sum_{\rho=0}^{n-n_0} \binom{n-\rho}{k}\Big) + f(n_0) = \\ &= \sum_{k=0}^{r} a_k \Big(\sum_{\lambda=n_0}^{n} \binom{\lambda}{k}\Big) + f(n_0) = \sum_{k=0}^{r} a_k \Big(\sum_{\sigma=n_0-k}^{n-k} \binom{k+\sigma}{k}\Big) + f(n_0) = \\ &= \sum_{k=0}^{r} a_k \Big(\sum_{\sigma=0}^{n-k} \binom{k+\sigma}{k} - \sum_{\sigma=0}^{n_0-k-1} \binom{k+\sigma}{k}\Big) + f(n_0) = \sum_{k=0}^{r} a_k \binom{n+1}{k+1} + c \end{aligned}$$

where $c \in \mathbb{Q}$ is a constant. The polynomial $P(t) = \sum_{k=0}^{r} a_k \binom{t}{k+1} + c \in \mathbb{Q}[t]$ has degree $\leq r+1$ and satisfies : $P(n) = f(n)$ for all $n > n_0$.

(4.37) <u>Proposition</u> and <u>Definition</u>: Let A be a semilocal Noetherian ring, $I \subseteq A$ an ideal of definition, and M a finitely generated A-module. There is a polynomial $\chi_M^I \in \mathbb{Q}[t]$ such that for all but finitely many $n \in \mathbb{N}$: $\chi_M^I(n) = \ell_A(M/I^{n+1}M)$.
χ_M^I is called the <u>Hilbert-Samuel polynomial</u> of M with respect to I. The degree of χ_M^I is denoted by $d = d(M)$.

<u>Proof</u>: The exact sequence: $0 \longrightarrow I^{n+1}M/I^{n+2}M \longrightarrow M/I^{n+2}M \longrightarrow M/I^{n+1}M \longrightarrow 0$
yields that : $\ell_A(M/I^{n+2}M) - \ell_A(M/I^{n+1}M) = \ell_A(I^{n+1}M/I^{n+2}M) = \varphi_M^I(n+1)$
for sufficiently large n. The statement follows by (4.36).

(4.38) <u>Proposition</u>: Let A be a semilocal Noetherian ring, $I, J \subseteq A$ ideals of definition, and M a finitely generated A-module. Then

$$\deg \chi_M^I = d(M) = \deg \chi_M^J .$$

The degree of the Hilbert-Samuel polynomial is independent of the choice of the ideal of definition.

<u>Proof</u>: Let $m = \mathrm{Jrad}(A)$ and $a, b \in \mathbb{N}$ such that $m^a \subseteq I \subseteq m$ and $m^b \subseteq J \subseteq m$. This implies $I^b \subseteq J$ and $J^a \subseteq I$ and thus for all $n \in \mathbb{N}$: $I^{b(n+1)}M \subseteq J^{n+1}M$ and $J^{a(n+1)}M \subseteq I^{n+1}M$.
Hence: $\ell_A(M/I^{b(n+1)}M) \geq \ell_A(M/J^{n+1}M)$ and $\ell_A(M/J^{a(n+1)}M) \geq \ell_A(M/I^{n+1}M)$.

Therefore for sufficiently large n:

$$\chi_M^I(bn+b-1) \geq \chi_M^J(n) \quad \text{and} \quad \chi_M^J(an+a-1) \geq \chi_M^I(n).$$

This implies:

$$\deg \chi_M^I(bn+b-1) = \deg \chi_M^I(t) \geq \deg \chi_M^J(t) = \deg \chi_M^J(at+a-1) \geq \deg \chi_M^I(t).$$

(4.39) Theorem: Let A be a semilocal Noetherian ring and $0 \to M' \to M \to M'' \to 0$ an exact sequence of finitely generated A-modules. Then $d(M) = \max(d(M'), d(M''))$. Moreover, if $I \subseteq A$ is an ideal of definition $\chi^I_{M'}$ and $\chi^I_M - \chi^I_{M''}$ have the same degree and the same leading coefficient.

Proof: We may assume that M' is a submodule of M and that $M'' = M/M'$. Let $I \subseteq A$ be an ideal of definition. Since $M''/I^nM'' \cong M/M'+I^nM$ the sequence

$$0 \longrightarrow M'+I^nM/I^nM \longrightarrow M/I^nM \longrightarrow M/M'+I^nM \longrightarrow 0$$

is exact. This yields:

$$\ell_A(M/I^nM) = \ell_A(M'+I^nM/I^nM) + \ell_A(M/M'+I^nM) = \ell_A(M'/M'\cap I^nM) + \ell_A(M''/I^nM'').$$

Let $\tilde{\varphi}: \mathbb{N} \to \mathbb{N}$ be a function with $\tilde{\varphi}(n) = \ell_A(M'/M'\cap I^nM)$ for all $n \in \mathbb{N}$. Then for all $n \geq n_0$: $\chi^I_M(n-1) = \chi^I_{M''}(n-1) + \tilde{\varphi}(n)$ and there is a polynomial $\varphi(t) \in \mathbb{Q}[t]$ with $\varphi(n) = \tilde{\varphi}(n)$ for all $n \geq n_0$. This shows that $d(M) = \deg \chi^I_M = \max(\deg \chi^I_{M''}, \deg \varphi)$.

By Artin-Rees (4.18) there is an integer $c \in \mathbb{N}$ such that for all $n \geq c$:

$$I^{n+1}M' \subseteq M' \cap I^{n+1}M \subseteq I^{n+1-c}M'.$$

This implies $\chi^I_{M'}(n) \geq \varphi(n+1) \geq \chi^I_{M'}(n-c)$ for all $n \geq \max(n_0, c)$. Hence $\deg \chi^I_{M'} = \deg \varphi$ and φ and $\chi^I_{M'}$ have the same leading coefficient.

(4.40) Corollary: Let A be a semilocal Noetherian ring, M a finitely generated A-module, and $M' \subseteq M$ a submodule. Then $d(M') \leq d(M)$ and $d(M/M') \leq d(M)$.

(4.41) Corollary: Let A be a semilocal Noetherian ring, M a finitely generated A-module, and $a \in A$ a NZD on M. Then $d(M/aM) < d(M)$.

Proof: Consider the exact sequence $0 \longrightarrow M \xrightarrow{\tau} M \longrightarrow M/aM \longrightarrow 0$ where $\tau(m) = am$ for all $m \in M$. Then $M' = \mathrm{im}(\tau) \cong M$ and $\chi^I_{M'} = \chi^I_M$. By (4.39) χ^I_M and $\chi^I_M - \chi^I_{M''}$ $(M'' = M/aM)$ have the same degree and the same leading coefficient. Thus $\deg \chi^I_{M''} < \deg \chi^I_M$.

§3: DIMENSION

(4.42) Definition: Let A be a semilocal Noetherian ring and M a nonzero finitely generated A-module. Define:

$$s_A(M) := \inf\{ n \in \mathbb{N} \mid \exists\, a_1, \dots, a_n \in \mathrm{Jrad}(A) \text{ so that } \ell_A(M/(a_1M + \dots + a_nM)) < \infty \}.$$

(4.43) Remark: Assumptions as in (4.43). If $\mathrm{Jrad}(A) = (a_1, \dots, a_r)$ then $\ell_A(M/\mathrm{Jrad}(A)M) < \infty$ and $s_A(M) \leq r$. In particular, $s_A(M) < \infty$.

Let A be a semilocal Noetherian ring, $I \subseteq A$ an ideal of definition, and M a finitely generated A-module. The Hilbert-Samuel polynomial $\chi_M^I(t) \in \mathbb{Q}[t]$ of M with respect to I satisfies for large n: $\chi_M^I(n) = \ell_A(M/I^{n+1}M)$.

By (4.38) $d_A(M) = \deg(\chi_M^I)$ does not depend on the choice of the ideal of definition I.
By (4.39) for an exact sequence of finitely generated A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

$d_A(M) = \max\{d_A(M'), d_A(M'')\}$ and $\deg(\chi_M^I - \chi_{M'}^I - \chi_{M''}^I) < d_A(M)$.
Our goal is to compare: $\dim(M)$, $d_A(M)$ and $s_A(M)$.

(4.44) Lemma: Let A be a semilocal Noetherian ring and M a nonzero finitely generated A-module. Then:

(a) For all $a \in \mathrm{Jrad}(A)$: $s_A(M/aM) \geq s_A(M) - 1$.

(b) If $a \in \mathrm{Jrad}(A)$ and $a \notin P$ for all minimal primes $P \in \mathrm{Supp}_A(M)$ then $d_A(M/aM) \leq d_A(M) - 1$.

Proof: Let $\overline{M} = M/aM$ and note that $\overline{M} \neq 0$ by Nakayama.
(a) Suppose that $s = s_A(\overline{M})$ and $a_1, \dots, a_s \in \mathrm{Jrad}(A)$ with $\ell_A(\overline{M}/(a_1, \dots, a_s)\overline{M}) < \infty$.
Since $\overline{M}/(a_1, \dots, a_s)\overline{M} \cong M/(a, a_1, \dots, a_s)M$ it follows that $s_A(\overline{M}) + 1 \geq s_A(M)$.
(b) Obviously, $\mathrm{ann}_A(M) \subseteq \mathrm{ann}_A(\overline{M})$. Let $\{P_1, \dots, P_r\}$ be the set of minimal prime

ideals of $\mathrm{Supp}_A(M) = V(\mathrm{ann}_A(M))$. By assumption $a \notin P_i$ for all $1 \le i \le r$. If $P \supseteq \mathrm{ann}_A(\bar{M})$ then $a \in P$ and $P_i \subsetneq P$ for some $1 \le i \le r$. This implies:

$$\dim(A/P) < \dim(A/P_i) \le \dim(A/\mathrm{ann}_A(M)) = \dim(M).$$

Thus $\dim_A(M/aM) \le \dim_A(M) - 1$.

(4.45) **Theorem**: Let A be a semilocal Noetherian ring and M a nonzero finitely generated A-module. Then $\dim_A(M) = d_A(M) = s_A(M)$.

Proof: (1) We may replace A by $\bar{A} = A/\mathrm{ann}(M)$ and assume that $\mathrm{ann}_A(M) = 0$.

Pf of (1): (a) Obviously, $\dim_A(M) = \dim_{\bar{A}}(M)$.

(b) $d_A(M) = d_{\bar{A}}(M)$. If $I \subseteq \mathrm{Jrad}(A)$ is an ideal of definition of A then $\bar{I} = I + \mathrm{ann}_A(M) \subseteq \bar{A}$ is an ideal of definition of $\bar{A}$ and $\ell_A(M/I^n M) = \ell_{\bar{A}}(M/\bar{I}^n M)$ and $\chi_M^I = \chi_M^{\bar{I}}$.

Thus $d_A(M) = d_{\bar{A}}(M)$.

(c) $s_A(M) = s_{\bar{A}}(M)$: Let $m\text{-}\mathrm{Spec}(A) = \{m_1, \ldots, m_n\}$. If $\mathrm{ann}_A(M) \subseteq m_i$ for all $1 \le i \le n$ then $s_A(M) = s_{\bar{A}}(M)$ since every $\bar{a} \in \mathrm{Jrad}(\bar{A})$ is image of an element $a \in \mathrm{Jrad}(A)$.

Suppose $\mathrm{ann}_A(M) \subseteq m_1 \cap \ldots \cap m_r$ and $\mathrm{ann}_A(M) \nsubseteq m_{r+1} \cup \ldots \cup m_n$. If $a_1, \ldots, a_s \in \mathrm{Jrad}(A)$ with $\ell_A(M/(a_1, \ldots, a_s)M) < \infty$ then $\bar{a}_1, \ldots, \bar{a}_s \in \mathrm{Jrad}(\bar{A})$ with $\ell_{\bar{A}}(M/(\bar{a}_1, \ldots, \bar{a}_s)M) < \infty$.

Thus $s_A(M) \ge s_{\bar{A}}(M)$. In order to show equality note that $\mathrm{ann}_A(M) + m_{r+1} \cap \ldots \cap m_n = A$ and let $q \in \mathrm{ann}_A(M)$ and $t \in m_{r+1} \cap \ldots \cap m_n$ with $q + t = 1$. Let $\bar{a}_1, \ldots, \bar{a}_s \in \mathrm{Jrad}(\bar{A})$ with $\ell_{\bar{A}}(M/(\bar{a}_1, \ldots, \bar{a}_s)M) < \infty$ and let $a_1, \ldots, a_s \in m_1 \cap \ldots \cap m_r$ be preimages of the a_i.

Then $a_i = a_i q + a_i t$ and $a_i M = a_i t M$ where $a_i t \in \mathrm{Jrad}(A)$ for all $1 \le i \le s$. Thus $\ell_{\bar{A}}(M/(\bar{a}_1, \ldots, \bar{a}_s)M) = \ell_A(M/(a_1, \ldots, a_s)M) = \ell_A(M/(a_1 t, \ldots, a_s t)M) < \infty$. This shows $s_A(M) \le s_{\bar{A}}(M)$ and we may assume that $\mathrm{ann}_A(M) = 0$.

(2) $\dim_A(M) \le d_A(M)$

Pf of (2): By induction on $d = d_A(M)$. Let $m = \mathrm{Jrad}(A)$. If $d_A(M) = 0$ then $\ell_A(M/m^{n+1}M) = \text{const.}$ for all $n \ge n_0$. Thus $m^{n+1}M = m^n M$ for $n \ge n_0$ and by Nakayama $m^n M = 0$. Thus $m^n \subseteq \mathrm{ann}_A(M)$ and by (1.85) A is an Artinian ring and $\dim(A) = 0$.

For the induction step suppose that $d_A(M) = d+1$ and let $P_0 \subseteq A$ be a prime ideal with $\dim(A/P_0) = \dim(A) = \dim_A(M)$. (Note that possibly $\dim(A) = \infty$.) Since $\mathrm{ann}_A(M) = (0)$ it holds that $\mathrm{Supp}_A(M) = \mathrm{Spec}(A)$. P_0 is a minimal prime ideal of A with $P_0 \in \mathrm{Ass}_A(M)$ and there is a submodule $N \subseteq M$ with $N \cong A/P_0$. By (4.39) $d_A(N) \leq d_A(M)$ and it suffices to show that $d_A(N) \geq \dim_A(N) = \dim_A(M)$. Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be a chain of prime ideals of A. We need to show that $n \leq d_A(N)$. If $n = 0$ we are done. If $n \geq 1$ pick $a \in P_1 - P_0$. Since $N \cong A/P_0$, the element a is a NZD of N and $N/aN \cong A/P_0 + (a)$. Since $P_1/P_0+(a) \subsetneq P_2/P_0+(a) \subsetneq \cdots \subsetneq P_n/P_0+(a)$ is a chain of length $n-1$ in $A/P_0+(a)$ we obtain that $\dim_A(N/aN) = \dim_A(A/P_0+(a)) \geq n-1$.

By (4.41): $d_A(N/aN) \leq d_A(N) - 1 < d_A(M)$ the induction hypothesis applies to N/aN and $\quad d_A(N) - 1 \geq d_A(N/aN) \geq \dim_A(N/aN) \geq n-1$.

Thus $d_A(N) \geq n$ and $d_A(M) \geq \dim_A(M)$.

Note that (2) shows that $\dim_A(M) < \infty$.

(3) $d_A(M) \leq s_A(M)$

Pf of (3): Suppose $s = s_A(M)$ and let $a_1, \ldots, a_s \in \mathfrak{J}\mathrm{rad}(A)$ with $\ell_A(M/(a_1,\ldots,a_s)M) < \infty$. Let $I = (a_1, \ldots, a_s)$. We want to show that I is an ideal of definition or equivalently that A/I is an Artinian ring. Let $P \in \mathrm{Spec}(A)$ with $I \subseteq P$. Since $\mathrm{ann}_A(M) = 0$ the localization $M_P \neq 0$ and by Nakayama $(M/IM)_P = M_P/IM_P \neq 0$. Thus $P \in \mathrm{Supp}_A(M/IM)$. Let $Q \subseteq P$ be a minimal prime in $\mathrm{Supp}_A(M/IM)$. Then $Q \in \mathrm{Ass}_A(M/IM)$ and there is a submodule $N \subseteq M/IM$ with $N \cong A/Q$. Since $\ell_A(M/IM) < \infty$, $\ell_A(N) = \ell_A(A/Q) = \ell_{A/Q}(A/Q) < \infty$. By (1.81) A/Q satisfies the d.c.c. and A/Q is an Artinian ring. By (1.82) every prime ideal of A/Q is maximal. Thus $P = Q$ and P is a maximal ideal of A. $\mathrm{rad}(I) = \mathfrak{J}\mathrm{rad}(A)$ and A/I is an Artinian ring. By (4.30) $\deg \varphi_M^I \leq s-1$ and therefore $d_A(M) = \deg(\chi_M^I) \leq s$.

(4) $s_A(M) \leq \dim_A(M)$

Pf of (4): By (2): $\dim_A(M) < \infty$. The proof is by induction on $n = \dim_A(M) = \dim(A)$.

If $n=0$ then A is Artinian and $\ell_A(M)<\infty$. Thus $s_A(M)=0$.

For the induction step suppose that $\dim_A(M)=n+1$. Let $P_1,\ldots,P_r$ be the minimal prime ideals of A with $\dim(A/P_i)=\dim(A)=\dim_A(M)$ for $1\le i\le r$. Since $\dim_A(M)\ge 1$, none of the P_i is maximal in A and $m=\operatorname{Jrad}(A)\not\subseteq\bigcup_{i=1}^r P_i$. Let $a\in m-\bigcup_{i=1}^r P_i$.

By (4.44)(a): $s_A(M/aM)\ge s_A(M)-1$.

By the construction of a: $\dim(A/aA)\le\dim(A)-1=\dim_A(M)-1$.

Since $\dim_A(M/aM)\le\dim(A/aA)\le\dim(A)-1=n$ the induction hypothesis applies to M/aM and $s_A(M)-1\le s_A(M/aM)\le\dim_A(M/aM)\le\dim(A/aA)\le\dim(A)-1=n$.

This shows: $s_A(M)\le n+1=\dim_A(M)$.

(4.46) <u>Corollary</u>: Let A be a semilocal Noetherian ring. The dimension $\dim(A)$ of A is the least number of generators of an ideal of definition of A.

(4.47) <u>Definition</u>: Let A be a local Noetherian ring with maximal ideal m and M a finitely generated A-module with $\dim_A(M)=n$. A set of n elements $\{a_1,\ldots,a_n\}\subseteq m$ is called a <u>system of parameters</u> of M if $\ell_A(M/(a_1,\ldots,a_n)M)<\infty$.

(4.48) <u>Remark</u>: Systems of parameters always exist.

(4.49) <u>Definition</u>: Let A be a ring and M a finitely generated A-module. The minimal number of generators of M is denoted by $\mu_A(M)$.

$$\mu_A(M)=\min\{s\in\mathbb{N}\mid \exists\ m_1,\ldots,m_s\in M \text{ such that } M=\sum_{i=1}^s Am_i\}.$$

In the following (A,m,k) denotes a local ring A with maximal ideal m and residue field $A/m=k$.

(4.50) <u>Lemma</u>: Let (A,m,k) be a local ring (not necessarily Noetherian) and M a finitely generated A-module. Then $\mu_A(M)=\dim_k(M/mM)$.

<u>Proof</u>: Lemma of Nakayama.

(4.51) <u>Corollary</u>: Let (A, m, k) be a local Noetherian ring. Then $\dim(A) \leq \dim_k(m/m^2)$.

<u>Proof</u>: By (4.45) and (4.50)

(4.52) <u>Definition</u>: Let (A, m, k) be a local Noetherian ring.

(a) $\dim_k(m/m^2) = \operatorname{edim}(A)$ is called the <u>embedding dimension</u> of A.

(b) A is called a <u>regular local ring</u> (RLR) if $\dim(A) = \operatorname{edim}(A) = \dim_k(m/m^2)$.

(4.53) <u>Theorem</u>: Let A be a Noetherian ring and $P \subseteq A$ a prime ideal. The following conditions are equivalent:

(a) $\operatorname{ht} P \leq n$

(b) There are elements $a_1, \ldots, a_n \in A$ such that P is a minimal prime over $(a_1, \ldots, a_n)$.

<u>Proof</u>: A_P is a local Noetherian ring with maximal ideal PA_P and $\dim A_P = \operatorname{ht} P$.

(a) $\Rightarrow$ (b): Since $\operatorname{ht} P = \dim A_P \leq n$ by (4.45) there are $s \in A - P$ and $a_1, \ldots, a_n \in P$ such that $(\frac{a_1}{s}, \ldots, \frac{a_n}{s})A_P$ is PA_P-primary. P is a minimal prime over $(a_1, \ldots, a_n)$.

(b) $\Rightarrow$ (a) Let $I = (a_1, \ldots, a_n)$ and P minimal over I. Then IA_P is PA_P-primary and IA_P is an ideal of definition of A_P. By (4.45) $\dim A_P = \operatorname{ht} P \leq n$.

(4.54) <u>Corollary</u>: (Krull's principal ideal theorem) Let A be a Noetherian ring and $a \in A - A^*$ a nonunit of A. Every minimal prime ideal over aA has height ≤ 1.

(4.55) <u>Corollary</u>: (Generalized principal ideal theorem) Let A be a Noetherian ring and $I = (a_1, \ldots, a_n) \subsetneq A$ an ideal. Every minimal prime ideal over I has height $\leq n$, in particular, $\operatorname{ht} I \leq n$.

(4.56) Corollary: Let A be a Noetherian ring and $P_0 \subsetneq P_1 \subsetneq P_2$ a chain of prime ideals of A. There are infinitely many prime ideals $Q \in \operatorname{Spec}(A)$ with $P_0 \subsetneq Q \subsetneq P_2$.

Proof: We may assume that $P_0 = 0$ and that A is a local Noetherian domain with maximal ideal $m = P_2 A_{P_2}$. By assumption ht $m = \dim A \geq 2$. By (4.54) every element $a \in m - (0)$ is contained in a prime ideal $Q \subseteq A$ with ht $Q = 1$. Thus $m = \bigcup_{Q \text{ prime}, \operatorname{ht} Q = 1} Q$. If there are only finitely many prime ideals Q of height one then $m = Q$ for some height one prime ideal Q, a contradiction.

(4.57) Theorem: Let A be a Noetherian ring, $A[x_1, \ldots, x_n]$ the polynomial ring in n variables over A. Then $\dim A[x_1, \ldots, x_n] = \dim A + n$.

Obviously it suffices to show that $\dim A[x] = \dim A + 1$. The proof requires several Lemmas.

(4.58) Lemma: Let A be a Noetherian ring, x a variable over A, and $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_r$ a chain of prime ideals of A. $P_0 A[x] \subsetneq P_1 A[x] \subsetneq \ldots \subsetneq P_r A[x] \subsetneq P_r A[x] + (x)$ is a chain of prime ideals of $A[x]$.

Proof: $A[x]/P_i A[x] \cong (A/P_i)[x]$ and

$$A[x]/(P_r + (x))A[x] \cong \left(A[x]/P_r A[x]\right) / \left((P_r + (x))A[x]/P_r A[x]\right) \cong (A/P_r)[x] / x(A/P_r)[x] \cong A/P_r.$$

(4.59) Corollary: $\dim A[x] \geq \dim A + 1$

(4.60) Lemma: Let $P \subsetneq Q \subseteq A[x]$ be prime ideals with $P \cap A = Q \cap A$. Then $P = (P \cap A)A[x]$.

Proof: We may assume that $P \cap A = Q \cap A = 0 \in \operatorname{Spec}(A)$, that is, $P \cap A = 0$ and A is a domain. If $P \neq 0$, there is a chain of prime ideals in $A[x]$: $0 \subsetneq P \subsetneq Q$, where $P \cap A = Q \cap A = 0$. With $S = A - (0)$, $S^{-1}(A[x]) = Q(A)[x] = K[x]$ where $K = Q(A)$ is the quotient field of A. $0 \subsetneq PK[x] \subsetneq QK[x]$ are prime ideals in $K[x]$ and $\dim K[x] \geq 2$, a contradiction.

(4.61) <u>Lemma</u>: Let $I \subseteq A$ be an ideal and $P \in \operatorname{Spec}(A)$ a prime ideal which is minimal over I. $PA[x]$ is a prime ideal minimal over $IA[x]$.

<u>Proof</u>: By (4.60).

(4.62) <u>Lemma</u>: Let A be a Noetherian ring and $P \in \operatorname{Spec}(A)$. Then $\operatorname{ht} P = \operatorname{ht} PA[x]$.

<u>Proof</u>: Let $\operatorname{ht} P = n$. By (4.53) there are elements $a_1, \ldots, a_n \in A$ so that P is a minimal prime ideal over $I = (a_1, \ldots, a_n)$. By (4.61) $PA[x]$ is a minimal prime ideal over $IA[x]$. Thus $\operatorname{ht} PA[x] \leq n$.
Let $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n = P$ be a chain of prime ideals in A. Then $P_0A[x] \subsetneq P_1A[x] \subsetneq \ldots \subsetneq P_nA[x] = PA[x]$ is a chain of prime ideals in $A[x]$ and $\operatorname{ht} PA[x] \geq n$.

<u>Proof of (4.57)</u>: We have to show that $\dim A[x] \leq \dim A + 1$.
Let $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_r$ be a chain of prime ideals in $A[x]$ and let $P_i = Q_i \cap A$. Suppose that $P_i = P_{i+1}$ for some i and let j be maximal with $P_j = P_{j+1}$. By (4.60) $Q_j = P_jA[x]$ and by (4.62) $\operatorname{ht} P_j = \operatorname{ht} Q_j$. This shows that $\operatorname{ht} P_j \geq j$. $P_j = P_{j+1} \subsetneq P_{j+2} \subsetneq \ldots \subsetneq P_n$ is a chain of prime ideals in A of length $r-j-1$. Thus $\dim A \geq r-1$.

(4.63) <u>Corollary</u>: Let K be a field. Then
(a) $\dim K[x_1, \ldots, x_n] = n$
(b) For all $1 \leq r \leq n$: $\operatorname{ht}(x_1, \ldots, x_r) = r$.

<u>Proof</u>: (b) By (4.53): $\operatorname{ht}(x_1, \ldots, x_r) \leq r$. $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \ldots \subsetneq (x_1, \ldots, x_r)$ is a chain of prime ideals, thus $\operatorname{ht}(x_1, \ldots, x_r) = r$.