§3: THE KRULL-AKIZUKI THEOREM

First we need some more facts about Artinian rings and modules.

(5.45) Proposition: Let A be a Noetherian ring with $\mathrm{Spec}(A) = \mathrm{m\text{-}Spec}(A)$. Every finitely generated A-module has finite length, in particular, A has finite length.

Proof: Let M be a finitely generated A-module. By (2.16) there is a normal series of M: $(0) = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ with $M_i/M_{i-1} \cong A/P_i$ where $P_i \in \mathrm{Spec}(A)$ for all $1 \leq i \leq n$. Since $P_i \in \mathrm{m\text{-}Spec}(A)$ the modules M_i/M_{i-1} are simple.

(5.46) Proposition: Let A be a ring. The following are equivalent:

(a) A is Noetherian and every prime ideal of A is maximal.

(b) Every finitely generated A-module has finite length.

(c) A has finite length (as an A-module).

Proof: (a) $\Rightarrow$ (b): By (5.45) and (b) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (a): Suppose that A has finite length as an A-module. Then A is Noetherian and Artinian. By (1.82) every prime ideal of A is maximal.

(5.47) Theorem: Let A be a Noetherian ring and M a finitely generated A-module. The following are equivalent:

(a) $\ell_A(M) < \infty$

(b) M is an Artinian A-module

(c) If $P \in \mathrm{Ass}_A(M)$ then P is a maximal ideal of A.

(d) $A/\mathrm{ann}(M)$ is an Artinian ring.

Proof: (a) $\Leftrightarrow$ (b): By (1.81).

(a) $\Rightarrow$ (c): Consider a composition series of M: $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_{n-1} \subsetneq M_n = M$, that is, the factor modules M_i/M_{i-1} are simple A-modules. Thus $M_i/M_{i-1} \cong A/m_i$ where $m_i \subseteq A$ is a maximal ideal. By (2.10) $\mathrm{Ass}_A(M) \subseteq \{m_1, \dots, m_n\}$.

(c) $\Rightarrow$ (d): By (2.16) the minimal prime ideals of $\mathrm{Ass}_A(M)$ and of $\mathrm{Supp}_A(M)$ are the same. Thus $\mathrm{Ass}_A(M) = \{m_1, \dots, m_n\} = \mathrm{Supp}_A(M) = V(\mathrm{ann}(M)) \subseteq m\text{-}\mathrm{Spec}(A)$. Thus $\dim A/\mathrm{ann}(M) = 0$ and $A/\mathrm{ann}(M)$ is an Artinian ring.

(d) $\Rightarrow$ (a): Every finitely generated module over an Artinian ring is Artinian.

(5.48) Remark: Let A be a domain. The following are equivalent:

(a) A is Noetherian and $\dim A = 1$.

(b) For every ideal $I \subseteq A$ with $I \neq (0)$ the ring A/I is Artinian.

Proof: (b) $\Rightarrow$ (a): We need to show that every ideal $I \subseteq A$ is finitely generated. Let $I \subseteq A$ be an ideal with $I \neq (0)$ and let $a \in I, a \neq 0$. By assumption $A/(a)$ is Artinian. Thus $A/(a)$ is Noetherian and $I/(a)$ is finitely generated. This implies that I is finitely generated.

(5.49) Theorem: (Krull–Akizuki) Let A be a Noetherian domain of dimension one and let $Q(A) \subseteq L$ be a finite field extension. Every intermediate ring $A \subseteq B \subseteq L$ is either a field or a Noetherian ring of dimension one.

We need the following Lemma:

(5.50) Lemma: Let A be a ring as in (5.49), V a vector space over $Q(A)$ of dimension $s < \infty$, $M \subseteq V$ an A-submodule, and $a \in A - (0)$. Then $\ell_A(M/aM) \leq s \cdot \ell_A(A/(a))$.

Proof of (5.49): Let $J \subseteq B$ a nonzero ideal and $b \in J - (0)$. Since b is algebraic over $Q(A)$ there are elements $a_i \in A$, $a_r \neq 0$, such that: $a_r b^r + a_{r-1} b^{r-1} + \dots + a_0 = 0$.

We may assume $a_0 \neq 0$. Thus $a_0 \in Bb \subseteq J$. Apply (5.50) to $L = V$ and $M = B$ to obtain:

$$\ell_B(B/\mathfrak{J}) \le \ell_A(B/\mathfrak{J}) \le \ell_A(B/a_0B) \le [L:K]\,\ell_A(A/a_0A) < \infty$$

The statement follows with (5.48).

<u>Proof</u> of (5.50): We may assume that M generates the $Q(A)$-vector space V.

<u>1st case</u>: M is a finitely generated A-module, say $M = Am_1 + \dots + Am_t$.

Then $s \le t$ and after renumbering $m_1, \dots, m_s$ is a basis of the $Q(A)$-vector space V. Then $m_1, \dots, m_s$ is also a basis of the free A-submodule $U = Am_1 + \dots + Am_s \subseteq M$. For all $s+1 \le i \le t$ there are elements $\lambda_{ij} \in Q(A)$ so that $m_i = \sum_{j=1}^{s} \lambda_{ij} m_j$. Hence there is an element $b \in A - (0)$ with $bM \subseteq U$. The A-module M/U has a nontrivial annihilator: $b \in \operatorname{ann}_A(M/U)$. Since $\dim A = 1$ and $\operatorname{ann}_A(M/U) \neq (0)$ the ring $A/\operatorname{ann}(M/U)$ is Artinian. M/U is a finitely generated $A/\operatorname{ann}(M/U)$-module and therefore: $\ell_A(M/U) < \infty$.

Let $a \in A - (0)$. For all $n \ge 1$ there is an exact sequence:

$$0 \longrightarrow U/a^nU \longrightarrow M/a^nU \longrightarrow M/U \longrightarrow 0$$

Note that $U/a^nU \cong (A/a^nA)^s$ and therefore $\ell_A(U/a^nU) = s.\,\ell_A(A/a^nA) < \infty$. This implies:

$$(*) \qquad \ell_A(M/a^nM) \le \ell_A(M/a^nU) = \ell_A(M/U) + \ell_A(U/a^nU) < \infty .$$

Since a is a non zero divisor on M and U:

$$M/aM \cong aM/a^2M \cong \dots \cong a^{n-1}M/a^nM \quad \text{and} \quad U/aU \cong aU/a^2U \cong \dots \cong a^{n-1}U/a^nU .$$

From the exact sequences:

$$0 \to a^{n-1}M/a^nM \to M/a^nM \to M/a^{n-1}M \to 0 \quad \text{and} \quad 0 \to a^{n-1}U/a^nU \to U/a^nU \to U/a^{n-1}U \to 0$$

we obtain by induction on n:

$$\ell_A(M/a^nM) = n\,\ell_A(M/aM) \quad \text{and} \quad \ell_A(U/a^nU) = n\,\ell_A(U/aU) .$$

$(*)$ yields for all $n \in \mathbb{N}$:

$$n\,\ell_A(M/aM) \le \ell_A(M/U) + ns\,\ell_A(A/aA)$$

and therefore: $\ell_A(M/aM) \le (1/n)\,\ell_A(M/U) + s\,\ell_A(A/aA)$ for all $n \in \mathbb{N}$.

Since $\ell_A(M/U) < \infty$: $\ell_A(M/aM) \le s\,\ell_A(A/aA)$.

<u>2d case</u>: M arbitrary

Let $\{M_\tau\}_{\tau \in T}$ be the set of all finitely generated A-submodules of M. Then for all $\tau \in T$: $\qquad \ell_A(M_\tau/aM_\tau) \le s\,\ell_A(A/aA)$.

Thus for all $\tau \in T$:

$$\ell_A\left(M_\tau + aM/aM\right) = \ell_A\left(M_\tau/M_\tau \cap aM\right) \leq \ell_A\left(M_\tau/aM_\tau\right) \leq s\,\ell_A\left(A/aA\right).$$

Choose $\tau_0 \in T$ such that $\ell_A\left(M_{\tau_0} + aM/aM\right)$ is maximal. If $M_{\tau_0} + aM \neq M$ pick $z \in M - \left(M_{\tau_0} + aM\right)$ and consider $M_{\tau_1} = M_{\tau_0} + Az$. Then

$$M_{\tau_1} + aM = M_{\tau_0} + Az + aM \supsetneq M_{\tau_0} + aM$$

and $\quad (M_{\tau_1} + aM)/aM \supsetneq (M_{\tau_0} + aM)/aM$.

This implies that $\ell_A\left(M_{\tau_1} + aM/aM\right) \gneq \ell_A\left(M_{\tau_0} + aM/aM\right)$, a contradiction.

Thus $M = M_{\tau_0} + aM$ and $\ell_A\left(M/aM\right) = \ell_A\left(M_{\tau_0} + aM/aM\right) \leq s \cdot \ell_A\left(A/aA\right)$.

<u>Note</u>: The Krull–Akizuki theorem fails for Noetherian domains A of dimension ≥ 2.

Example: Let K be a field and $K[x,y] \subseteq V \subseteq K(x,y)$ where V is a rank 2 valuation ring.

§4: FINITENESS OF THE INTEGRAL CLOSURE

(5.51) Definition: Let $K \subseteq L$ be a finite extension of fields of degree $[L:K] = n < \infty$. For an element $\alpha \in L$ let $\varphi_\alpha : L \longrightarrow L$ denote the K-linear map $\varphi_\alpha(t) = \alpha t$ for all $t \in L$. We define:

(a) $P_{L/K}(\alpha, x) := \det(x - \varphi_\alpha)$ the characteristic polynomial of α.

(b) $N_{L/K}(\alpha) := \det(\varphi_\alpha)$ the norm of α

(c) $Tr_{L/K}(\alpha) := \operatorname{trace}(\varphi_\alpha)$ the trace of α

(5.52) Facts from field theory:

Let $K \subseteq L$ be a finite extension of fields with $[L:K] = n < \infty$.

(a) There is a unique intermediate field $K \subseteq K_s \subseteq L$ such that

(i) $K \subseteq K_s$ is a separable field extension.

(ii) $K_s \subseteq L$ is a purely inseparable field extension, that is, if $\operatorname{char}(K) = 0$ then $K_s = L$ and if $\operatorname{char}(K) = p > 0$ then $L^q \subseteq K_s$ with $q = p^r$ for some $r \in \mathbb{N}$. Note that $K_s = \{\alpha \in L \mid \alpha \text{ is separable over } K\}$.

(iii) $r_i = [L:K_s]$ is called the inseparable degree of the extension $K \subseteq L$ while $r_s = [K_s : K]$ is called the separable degree of L over K. Then $n = r_i r_s$.

(iv) There are exactly r_s mutually distinct K-morphisms $\tau : K \longrightarrow \overline{L}$ where $\overline{K}$ is the algebraic closure of K (and L).

(b) Let $\tau_1, \ldots, \tau_{r_s} : L \longrightarrow \overline{K}$ be the r_s mutually distinct K-morphisms from L into $\overline{K}$ with $\tau_1 = id_L$ the canonical embedding. For an element $\alpha \in L$ the elements $\tau_j(\alpha) = \alpha^{(j)}$, $1 \leq j \leq s$, are called the conjugates of α. Note that $\alpha = \alpha^{(1)}$ and that possibly $\alpha^{(j)} = \alpha^{(k)}$ for $j \neq k$.

(i) $$P_{L/K}(\alpha, x) = \prod_{j=1}^{r_s} (x - \alpha^{(j)})^{r_i} \in K[x]$$

(ii) $$N_{L/K}(\alpha) = \prod_{j=1}^{r_s} \alpha^{(j)\, r_i} \in K$$

(iii) $$Tr_{L/K}(\alpha) = r_i \sum_{j=1}^{r_s} \alpha^{(j)} \in K, \text{ that is,}$$

$Tr_{L/K}(\alpha) = 0$ if $r_i > 1$ or equivalently if L is inseparable over K. Moreover, $Tr_{L/K}(\alpha) = \sum_{j=1}^{r_s} \alpha^{(j)}$ if $r_i = 1$ or equivalently if L is separable over K.

If $[L:K]=n$, then $\deg P_{L/K}(\alpha,x)=n$ with $P_{L/K}(\alpha,x)=x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_0$ where $a_{n-1}=Tr_{L/K}(\alpha)$ and $a_0 = N_{L/K}(\alpha)$. Norm and trace are elements of K.

(c) (i) The norm function: $N_{L/K} : L^* \longrightarrow K^*$ is a homomorphism of the multiplicative groups.
$$\alpha \longmapsto N_{L/K}(\alpha)$$
(ii) The trace function: $T_{L/K} : L \longrightarrow K$ is a K-linear transformation.
$$\alpha \longmapsto Tr_{L/K}(\alpha)$$

(5.53) <u>Definition:</u> Same assumptions as in (5.52). Let $\alpha \in L$ be an element.

(a) Let $P'_{L/K}(\alpha,x)$ denote the formal derivative of the polynomial $P_{L/K}(\alpha,x) \in K[x]$. The element $\delta_{L/K}(\alpha) = P'_{L/K}(\alpha,\alpha) \in L$ is called the <u>different</u> of α <u>over</u> L.

(b) The element
$$D(\alpha) = (-1)^{\frac{n(n-1)}{2}} N_{L/K}(\delta_{L/K}(\alpha)) \in K$$
is called the <u>discriminant</u> of α over L.

(5.54) <u>Lemma</u>: With the assumptions and notations of (5.52) for an element $\alpha \in L$ the following conditions are equivalent:

(a) $D(\alpha) \neq 0$

(b) $\delta_{L/K}(\alpha) \neq 0$

(c) L is separable over K with $L = K(\alpha)$.

<u>Proof</u>: (a) $\Leftrightarrow$ (b): trivial

(b) $\Rightarrow$ (c): If $\delta_{L/K}(\alpha) \neq 0$ then α is a simple root of $P_{L/K}(\alpha,x)$. Since the characteristic polynomial divides a power of the minimal polynomial in this case $P_{L/K}(\alpha,x)$ is the minimal polynomial. Moreover, $P_{L/K}(\alpha,x)$ is a separable polynomial. Thus $[K(\alpha):K]=n$ and $L=K(\alpha)$ is separable over K.

(c) $\Rightarrow$ (b): $P_{L/K}(\alpha,x)$ is the minimal polynomial of α over K and $P_{L/K}(\alpha,x)$ has only simple roots. Thus $\delta_{L/K}(\alpha) \neq 0$. [Note that if $m(\alpha,x)$ is the minimal polynomial of α, then $m(\alpha,x) \mid P_{L/K}(\alpha,x)$ and $P_{L/K}(\alpha,x) \mid m(\alpha,x)^t$ for some $t \in \mathbb{N}$.]

◀

(5.55) <u>Lemma</u>: Let $K \subseteq L$ be a finite separable field extension, $[L:K]=n$, and $\alpha \in L$. Then:

$$D(\alpha) = (-1)^{\frac{n(n-1)}{2}} \prod_{1 \le k < \ell \le n} (\alpha^{(k)} - \alpha^{(\ell)})^2 = \det \begin{bmatrix} 1 & \alpha^{(1)} & \text{—} & \alpha^{(1)n} \\ 1 & \alpha^{(2)} & \text{—} & \alpha^{(2)n} \\ & & & \\ 1 & \alpha^{(n)} & \text{—} & \alpha^{(n)n} \end{bmatrix}$$

<u>Proof</u>: If $K \subseteq L$ is a finite separable field extension of degree n, then there are exactly n mutually distinct K-morphisms $\tau_i : L \longrightarrow \overline{K}$. $\alpha \in L$ has exactly n conjugates $\alpha^{(1)}, \text{—}, \alpha^{(n)}$ and we have: $P_{L/K}(\alpha, x) = \prod_{j=1}^{n} (x - \alpha^{(j)})$ with $\alpha^{(1)} = \alpha$ and $P'_{L/K}(\alpha,\alpha) = \prod_{j=2}^{n} (\alpha - \alpha^{(j)}) = \delta_{L/K}(\alpha)$.

For $1 \le j \le n$:

$$\delta_{L/K}^{(j)}(\alpha) = \prod_{k=1, k \ne j}^{n} (\alpha^{(j)} - \alpha^{(k)}); \quad \text{and}$$

$$N_{L/K}(\delta_{L/K}(\alpha)) = \prod_{j=1}^{n} \prod_{k=1, k \ne j}^{n} (\alpha^{(j)} - \alpha^{(k)}) = (-1)^{\frac{n(n-1)}{2}} \prod_{1 \le k < j \le n} (\alpha^{(k)} - \alpha^{(j)})^2$$

Note that the double product has exactly $n(n-1)$ factors. Collecting factors $\alpha^{(k)} - \alpha^{(j)}$ and $\alpha^{(j)} - \alpha^{(k)}$ gives the formula. The rest is the Vandermond determinant from linear algebra.

(5.56) <u>Definition</u>: Let $K \subseteq L$ be a finite separable field extension with $[L:K]=n$. For elements $\omega_1, \text{—}, \omega_n \in L$ define the <u>discriminant</u> of $\omega_1, \text{—}, \omega_n$ by:

$$\Delta_{L/K}(\omega_1, \text{—}, \omega_n) = \det\left((\omega_i^{(j)})_{1 \le i,j \le n} \right)^2$$

where $\omega_i^{(1)}, \text{—}, \omega_i^{(n)}$ are the conjugates of ω_i.

(5.57) <u>Lemma</u>: With assumptions as in (5.56):

(a) $\Delta_{L/K}(\omega_1, \text{—}, \omega_n) = \det(\mathrm{Tr}(\omega_i \omega_j)_{1 \le i,j \le n})$

(b) $\Delta_{L/K}(\omega_1, \text{—}, \omega_n) \ne 0$ if and only if $\omega_1, \text{—}, \omega_n$ is a basis of the K-vector space L.

<u>Proof</u>: (a): $\Delta_{L/K}(\omega_1, \text{—}, \omega_n) = \det((\omega_i^{(j)}))^2 = \det((\omega_i^{(j)}))\det((\omega_i^{(j)})^T) =$

$$= \det\left(\left(\sum_{k=1}^{n} \omega_i^{(k)} \omega_j^{(k)} \right)_{i,j} \right) = \det\left(\mathrm{Tr}_{L/K}(\omega_i \omega_j)_{i,j} \right).$$

(b) Since $K \subseteq L$ is a separable extension, $L = K(\alpha)$ for some $\alpha \in L$. Thus $\omega_j = \sum_{i=1}^{n} a_{ij}\alpha^{i-1}$ for some $a_{ij} \in K$ and $\omega_j^{(k)} = \sum_{i=1}^{n} a_{ij}\alpha^{(k)\,i-1}$ for all $1 \le j \le n$ and $1 \le k \le n$.

Thus:
$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = (a_{ij})_{i,j} \begin{bmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{n-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \\ \vdots \\ \omega_n^{(k)} \end{bmatrix} = (a_{ij})_{i,j} \begin{bmatrix} 1 \\ \alpha^{(k)} \\ \vdots \\ \alpha^{(k)\,n-1} \end{bmatrix}$$

This implies:
$$\left(\omega_j^{(k)}\right)_{j,k} = (a_{ij})_{i,j}\left(\alpha^{(k)\,i-1}\right)_{i,k}$$

implying
$$\det\left(\omega_j^{(k)}\right) = \det\left((a_{ij})\right)\det\left(\alpha^{(k)\,i-1}\right)$$

and therefore by (3.55): $\Delta_{L/K}(\omega_1, \ldots, \omega_n) = \det((a_{ij}))^2 D(\alpha)$.

By (5.54) $D(\alpha) \neq 0$. This shows that $\Delta_{L/K}(\omega_1, \ldots, \omega_n) \neq 0$ if and only if $\det((a_{ij})) \neq 0$ which is equivalent to $\omega_1, \ldots, \omega_n$ a basis of L over K.

(5.58) <u>Corollary</u>: Let $K \subseteq L$ be a finite field extension. L is separable over K if and only if $\mathrm{Tr}_{L/K} \neq 0$ (i.e. the trace is not the zero function).

<u>Proof</u>: If $K \subseteq L$ is inseparable then by (5.52) $\mathrm{Tr}_{L/K} = 0$ (if $n > 1$).
Suppose that $K \subseteq L$ is separable and $\omega_1, \ldots, \omega_n$ is a basis of L over K. By (5.57) $\Delta_{L/K}(\omega_1, \ldots, \omega_n) = \det\left(\mathrm{Tr}_{L/K}(\omega_i\omega_j)\right) \neq 0$. Thus $\mathrm{Tr}_{L/K} \neq 0$.

(5.59) <u>Remark</u>: If $[L:K] = n$ then $\mathrm{Tr}_{L/K}(1) = n.1$. Thus if $\mathrm{char}(K) = 0$ then trivially $\mathrm{Tr}_{L/K} \neq 0$.

For the remainder of this section let $K \subseteq L$ be a finite field extension of degree $[L:K] = n$, A a Noetherian normal domain with field of quotients $K = Q(A)$ and B the integral closure of A in L.

(5.60) <u>Note</u>: (a) B is a normal domain.
(b) $B \cap K = A$

(c) There is a basis $\omega_1, \dots, \omega_n$ of the K-vector space L with $\omega_1, \dots, \omega_n \in B$.

(d) $L = K.B = Q(B)$.

(5.61) <u>Theorem</u>: Suppose there is a K-linear map $s: L \longrightarrow K$ with:

(a) $s(B) \subseteq A$

(b) $s \neq 0$

then B is a finite A-module.

<u>Proof</u>: Let $\omega_1, \dots, \omega_n \in B$ be a K-basis of L. Consider $M = \sum_{i=1}^{n} A\omega_i \subseteq B$. Since $\omega_1, \dots, \omega_n$ is a basis of L over K, M is a free A-module of rank n, that is, $M \cong A^n$. For any A-submodule $\mathcal{U}$ of L put:

$$\tilde{\mathcal{U}} := \{x \in L \mid s(x\mathcal{U}) \subseteq A\}.$$

$\tilde{\mathcal{U}}$ is an A-submodule of L and if $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq L$ are submodules of L then $\tilde{\mathcal{U}}_2 \subseteq \tilde{\mathcal{U}}_1$. Therefore, since $M \subseteq B$ we have $\tilde{B} \subseteq \tilde{M}$.

Since B is a ring for every $b \in B$: $bB \subseteq B$ and hence $s(bB) \subseteq s(B) \subseteq A$ by (a). This shows that $B \subseteq \tilde{B} \subseteq \tilde{M}$. Since A is Noetherian it suffices to show that $\tilde{M}$ is a finitely generated A-module. Consider the map:

$$\begin{aligned} \rho: \tilde{M} &\longrightarrow A^n \\ m &\longmapsto (s(m\omega_1), \dots, s(m\omega_n)) \end{aligned}$$

Since s is K-linear, ρ is an A-linear map.

<u>Claim</u>: ρ is injective

<u>Pf of Cl</u>: $\rho(m) = 0$ if and only if $s(m\omega_i) = 0$ for all $1 \le i \le n$. Hence if $\rho(m) = 0$ then $s(mM) = 0$. If $m \neq 0$ then $L = mL = K.mL$, since $m\omega_1, \dots, m\omega_n$ is a basis of L over K. Thus $m \neq 0$ and $s(mM) = 0$ implies that $s(L) = 0$ since s is K-linear. This contradicts assumption (b).

By the claim, $\tilde{M}$ and B can be considered A-submodules of A^n. Since A is Noetherian B is a finitely generated A-module.

(5.62) <u>Corollary</u>: Let $K \subseteq L$ be a finite separable field extension and $A \subseteq K$ a normal Noetherian domain with field of quotients $K = Q(A)$. The integral closure B of A in L is a finitely generated A-module. In particular, B is a Noetherian ring.

<u>Proof</u>: The trace function $Tr_{L/K} = s : L \to K$ satisfies conditions (a) and (b) of (5.61).

(5.63) <u>Corollary</u>: Let A be a Dedekind domain, $K = Q(A)$ its field of quotients, and $K \subseteq L$ a finite separable field extension. The integral closure B of A in L is a finitely generated A-module and a Dedekind domain.

(5.64) <u>Corollary</u>: Let $\mathbb{Q} \subseteq L$ be a finite field extension. The integral closure B of $\mathbb{Z}$ in L is a finitely generated $\mathbb{Z}$-module and a Dedekind domain.

(5.65) <u>Remark</u>: Let A be a Noetherian domain of dimension one, $K = Q(A)$ its field of quotients and $K \subseteq L$ a finite field extension. By the Krull-Akizuki theorem the integral closure B of A in L is a Noetherian domain of dimension one. Since B is normal B is a Dedekind domain. There are examples where B is not a finitely generated A-module.

(5.66) <u>Remark</u>: Let k be a field, A a finitely generated k-algebra and a domain, $K = Q(A)$ its field of quotients, and $K \subseteq L$ a finite field extension. The integral closure of A in L is a finitely generated A-module. In particular, the integral closure of A in K is a finitely generated A-module (without proof).