CHAPTER VII: HOMOLOGICAL ALGEBRA II

§1: COMPLEXES

(7.1) Definition: A complex of A-modules $(C_\cdot, \partial_\cdot)$ is a sequence of A-modules and A-linear maps:

$$C_\cdot : \quad \cdots\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots\cdots$$

so that $\partial_i \partial_{i+1} = 0$ for all $i \in \mathbb{Z}$. $\partial_\cdot$ is called the differential of the complex. The homology of the complex $(C_\cdot, \partial_\cdot)$ is the sequence of A-modules $H_i(C_\cdot) = \ker \partial_i / \operatorname{im} \partial_{i+1}$. The cycles $Z_\cdot(C_\cdot)$ and boundaries $B_\cdot(C_\cdot)$ are the sequences of A-modules $Z_i(C_\cdot) = \ker \partial_i$ and $B_i(C_\cdot) = \operatorname{im} \partial_{i+1}$.

(7.2) Remark: (a) $(C_\cdot, \partial_\cdot)$ is exact if and only if $H_i(C_\cdot) = 0$ for all $i \in \mathbb{Z}$.

(b) In order to avoid negative indices one often writes $(C^\cdot, \partial^\cdot)$ for:

$$C^\cdot : \quad \cdots\cdots \longrightarrow C^{i-1} = C_{-i+1} \xrightarrow{\partial^{i-1}} C^i = C_{-i} \xrightarrow{\partial^i} C^{i+1} = C_{-i-1} \longrightarrow \cdots$$

and $H^\cdot(C^\cdot)$ for the sequence $H^i(C^\cdot) = H_{-i}(C_\cdot)$. $H^\cdot(C^\cdot)$ is called the cohomology of $(C^\cdot, \partial^\cdot)$.

(7.3) Definition: (a) A morphism of complexes $u_\cdot : C_\cdot \longrightarrow C'_\cdot$ is a sequence of A-linear maps $u_i : C_i \longrightarrow C'_i$ so that $u_i \partial_{i+1} = \partial'_{i+1} u_{i+1}$ for all $i \in \mathbb{Z}$, that is, for all $i \in \mathbb{Z}$ the diagram:

$$\begin{tikzcd} C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "u_{i+1}"'] & C_i \arrow[d, "u_i"] \\ C'_{i+1} \arrow[r, "\partial'_{i+1}"] & C'_i \end{tikzcd}$$

commutes.

(b) A sequence of morphisms of complexes $0 \longrightarrow C'_\cdot \xrightarrow{u_\cdot} C_\cdot \xrightarrow{v_\cdot} C''_\cdot \longrightarrow 0$ is exact if $0 \longrightarrow C'_i \xrightarrow{u_i} C_i \xrightarrow{v_i} C''_i \longrightarrow 0$ is exact for all $i \in \mathbb{Z}$.

(c) The direct sum $C_\cdot \oplus C'_\cdot$ of two complexes $(C_\cdot, \partial_\cdot)$ and $(C'_\cdot, \partial'_\cdot)$ is the complex with $(C_\cdot \oplus C'_\cdot)_i = C_i \oplus C'_i$ and $\partial_i^{C_\cdot \oplus C'_\cdot} = \partial_i \oplus \partial'_i$.

(7.4) Remark: Let $u_\cdot : C_\cdot \longrightarrow C'_\cdot$ be a morphism of complexes, then for all $i \in \mathbb{Z}$ $u_i(Z_i(C_\cdot)) \subseteq Z_i(C'_\cdot)$ and $u_i(B_i(C_\cdot)) \subseteq B_i(C'_\cdot)$. Thus $u_\cdot$ induces a sequence of A-linear maps $H_\cdot(u_\cdot) : H_\cdot(C_\cdot) \longrightarrow H_\cdot(C'_\cdot)$ given by $H_i(u_\cdot)(z + B_i) = u_i(z) + B'_i$, where

$z \in Z_i(C_\cdot)$, $B_i = B_i(C_\cdot)$ and $B_i' = B_i(C_\cdot')$.

(7.5) <u>Theorem</u>: (Snake Lemma) Let

$$\begin{tikzcd}
 & M' \arrow[r, "\varphi"] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' &
\end{tikzcd}$$

be a commutative diagram with exact rows. Then there is a long exact sequence of induced maps:

$$\ker f' \xrightarrow{\overline{\varphi}} \ker f \longrightarrow \ker f'' \longrightarrow \operatorname{coker} f' \longrightarrow \operatorname{coker} f \xrightarrow{\overline{\psi}} \operatorname{coker} f''.$$

Moreover, if φ is injective then so is $\overline{\varphi}$, if ψ is surjective so is $\overline{\psi}$.

<u>Proof</u>: Diagram chasing:

$$\begin{tikzcd}
 & \ker f' \arrow[r, "\overline{\varphi}"] \arrow[d] & \ker f \arrow[r] \arrow[d] & \ker f'' \arrow[d] \arrow[ddddll, dashed, rounded corners] & \\
 & M' \arrow[r, "\varphi"] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r, "\psi"] \arrow[d] & N'' \arrow[d] & \\
 & \operatorname{coker} f' \arrow[r] & \operatorname{coker} f \arrow[r] & \operatorname{coker} f'' &
\end{tikzcd}$$

(7.6) <u>Proposition</u>: Let $0 \to C_\cdot' \xrightarrow{u_\cdot} C_\cdot \xrightarrow{v_\cdot} C_\cdot'' \to 0$ be an exact sequence of complexes.

(a) For every i there is an A-linear map (called <u>connecting homomorphism</u>)

$$\begin{aligned}
\Delta_i : H_i(C_\cdot'') &\longrightarrow H_{i-1}(C_\cdot') \qquad \text{given by} \\
z'' + B_i'' &\longmapsto u_{i-1}^{-1}\, \partial_i\, v_i^{-1}(z'') + B_{i-1}'
\end{aligned}$$

(b) There is an exact sequence of A-modules (called <u>long exact sequence of homology</u>):

$$\cdots \longrightarrow H_i(C_\cdot') \xrightarrow{H_i(u_\cdot)} H_i(C_\cdot) \xrightarrow{H_i(v_\cdot)} H_i(C_\cdot'') \xrightarrow{\Delta_i} H_{i-1}(C_\cdot') \xrightarrow{H_{i-1}(u_\cdot)} H_{i-1}(C_\cdot) \longrightarrow \cdots$$

(c) (Naturality of $\Delta_\cdot$) Let

$$\begin{tikzcd}
0 \arrow[r] & C_\cdot' \arrow[r, "u_\cdot"] \arrow[d, "f_\cdot'"] \arrow[dr, phantom, "\circlearrowleft"] & C_\cdot \arrow[r, "v_\cdot"] \arrow[d, "f_\cdot"] \arrow[dr, phantom, "\circlearrowleft"] & C_\cdot'' \arrow[r] \arrow[d, "f_\cdot''"] & 0 \\
0 \arrow[r] & D_\cdot' \arrow[r] & D_\cdot \arrow[r] & D_\cdot'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of morphisms of complexes with exact rows. Then the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(C'_\cdot) \arrow[r] \arrow[d, "H_i(f'_\cdot)"] \arrow[dr, phantom, "\circlearrowleft"] & H_i(C_\cdot) \arrow[r] \arrow[d, "H_i(f_\cdot)"] \arrow[dr, phantom, "\circlearrowleft"] & H_i(C''_\cdot) \arrow[r, "\Delta_i"] \arrow[d, "H_i(f''_\cdot)"] \arrow[dr, phantom, "\circlearrowleft"] & H_{i-1}(C'_\cdot) \arrow[r] \arrow[d, "H_{i-1}(f'_\cdot)"] \arrow[dr, phantom, "\circlearrowleft"] & H_{i-1}(C_\cdot) \arrow[r] \arrow[d, "H_{i-1}(f_\cdot)"] & \cdots \\
\cdots \arrow[r] & H_i(D'_\cdot) \arrow[r] & H_i(D_\cdot) \arrow[r] & H_i(D''_\cdot) \arrow[r, "\tilde{\Delta}_i"] & H_{i-1}(D'_\cdot) \arrow[r] & H_{i-1}(D_\cdot) \arrow[r] & \cdots
\end{tikzcd}$$

commutes and has exact rows.

Proof: By the Snake Lemma (7.5) the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C'_{i+1} \arrow[r] \arrow[d, "\partial'_{i+1}"] \arrow[dr, phantom, "\circlearrowleft"] & C_{i+1} \arrow[r] \arrow[d, "\partial_{i+1}"] \arrow[dr, phantom, "\circlearrowleft"] & C''_{i+1} \arrow[r] \arrow[d, "\partial''_{i+1}"] & 0 \\
0 \arrow[r] & C'_i \arrow[r] & C_i \arrow[r] & C''_i \arrow[r] & 0
\end{tikzcd}$$

with exact rows induces an exact sequence:

$$C'_i/B'_i \xrightarrow{\overline{u}_i} C_i/B_i \xrightarrow{\overline{v}_i} C''_i/B''_i \longrightarrow 0$$

where $B'_\cdot, B_\cdot, B''_\cdot$ denote boundaries. Likewise, again by the Snake Lemma, there is an induced exact sequence of cycles: $\quad 0 \longrightarrow Z'_{i-1} \xrightarrow{\overline{u}_{i-1}} Z_{i-1} \xrightarrow{\overline{v}_{i-1}} Z''_{i-1}$.

The differentials $\partial'_i, \partial_i, \partial''_i$ induce a commutative diagramm with exact rows:

$$\begin{tikzcd}
& C'_i/B'_i \arrow[r, "\overline{u}_i"] \arrow[d, "\overline{\partial}'_i"] \arrow[dr, phantom, "\circlearrowleft"] & C_i/B_i \arrow[r, "\overline{v}_i"] \arrow[d, "\overline{\partial}_i"] \arrow[dr, phantom, "\circlearrowleft"] & C''_i/B''_i \arrow[r] \arrow[d, "\overline{\partial}''_i"] & 0 \\
0 \arrow[r] & Z'_{i-1} \arrow[r, "\overline{u}_{i-1}"] & Z_{i-1} \arrow[r, "\overline{v}_{i-1}"] & Z''_{i-1} &
\end{tikzcd}$$

Note that $\ker \overline{\partial}_i = H_i(C_\cdot)$ and $\operatorname{coker} \overline{\partial}_i = H_{i-1}(C_\cdot)$ (likewise for $\overline{\partial}'_i$ and $\overline{\partial}''_i$). Then (a) and (b) are an immediate consequence of the Snake Lemma, and (c) is easy to see.

(7.7) Definition: a morphism of complexes $u_\cdot : C_\cdot \longrightarrow C'_\cdot$ is called <u>nullhomotopic</u> if there exists a sequence of A-linear maps $s_i : C_i \longrightarrow C'_{i+1}$:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[dl, "s_i"'] \arrow[d, "u_i"] & C_{i-1} \arrow[r] \arrow[dl, "s_{i-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{i+1} \arrow[r, "\partial'_{i+1}"] & C'_i \arrow[r, "\partial'_i"] & C'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

so that $u_i = \partial'_{i+1} s_i + s_{i-1}\partial_i$. Notation: $u_\cdot \sim 0$. Two morphisms of complexes $u_\cdot$ and $v_\cdot$ are <u>homotopic</u>, $u_\cdot \sim v_\cdot$, if $u_\cdot - v_\cdot \sim 0$. The sequence of maps s_i is called <u>homotopy</u>.

(7.8) Proposition: Let $u_\cdot, v_\cdot$ be morphisms of complexes $C_\cdot \longrightarrow C'_\cdot$. If $u_\cdot \sim v_\cdot$ then $H_\cdot(u_\cdot) = H_\cdot(v_\cdot)$.

Proof: If $u_\cdot \sim 0$, we need to show $H_\cdot(u_\cdot) = 0$. Let $z \in Z_i(C_\cdot)$. Then $u_i(z) + B'_i = \partial'_{i+1} s_i(z) + s_{i-1}\partial_i(z) + B'_i = B'_i$.

(7.9) Example: (a) A complex $C_\cdot$ is said to have contracting homotopy if $\mathrm{id}_{C_\cdot} \sim 0$. Notice that such a complex is exact.

(b)
$$\begin{array}{cccccccc} C_\cdot : & & 0 \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow 0 \\ u_\cdot \downarrow & & & \mathrm{id}\downarrow & & 0\downarrow & \\ C'_\cdot : & & 0 \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \end{array}$$
Notice that $H_\cdot(u_\cdot) = 0$ but $u_\cdot \not\sim 0$.

(7.10) Definition: (a) A complex $C_\cdot : \ldots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$ ($C^\cdot : 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \ldots$) is called acyclic if $H_i(C_\cdot) = 0$ ($H^i(C^\cdot) = 0$) for all $i \neq 0$.

(b) A projective resolution of a module M is an acyclic complex $P_\cdot$ with P_i projective modules for every i together with an isomorphism $H_0(P_\cdot) \cong M$ (or equivalently: $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is exact and P_i is projective for all i).

(c) An injective resolution of a module M is an acyclic complex $I^\cdot$ with I^i injective modules for every i together with an isomorphism $H^0(I^\cdot) \cong M$ (or equivalently: $0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots$ is exact and I^i is injective for all i).

(7.11) Remark: Every module has a projective and an injective resolution.

(7.12) Proposition: (a) Let $C_\cdot : \ldots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$ and $C'_\cdot : \ldots \longrightarrow C'_1 \longrightarrow C'_0 \longrightarrow 0$ be complexes where C_i are projective modules for all i and $C'_\cdot$ is acyclic. Then for every A-linear map $\varphi : H_0(C_\cdot) \longrightarrow H_0(C'_\cdot)$ there is a morphism of complexes $u_\cdot : C_\cdot \longrightarrow C'_\cdot$ with $H_0(u_\cdot) = \varphi$. Moreover, $u_\cdot$ is unique up to homotopy.

(b) Let $C^\cdot : 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \ldots$ and $C'^\cdot : 0 \longrightarrow C'^0 \longrightarrow C'^1 \longrightarrow \ldots$ be complexes where $C^\cdot$ is acyclic and C'^i are injective modules for all i. Then for every A-linear map

$\varphi: H^\circ(C^\cdot) \longrightarrow H^\circ(C'^\cdot)$ there is a morphism of complexes $u^\cdot: C^\cdot \longrightarrow C'^\cdot$ with $H^\circ(u_\cdot) = \varphi$.
Moreover, $u^\cdot$ is unique up to homotopy.

<u>Proof</u>: (a) Existence: We construct u_i inductively. For $i=0$, we have:

$$\begin{tikzcd} C_0 \arrow[r, "\pi"] \arrow[d, dashed, "u_0"'] \arrow[dr, "\varphi\pi"] & H_0(C_\cdot) \arrow[d, "\varphi"] \\ C_0' \arrow[r, "\pi'"] & H_0(C_0') \end{tikzcd}$$

$\varphi\circ\pi$ can be lifted to an A-linear map $u_0: C_0 \longrightarrow C_0'$ since π' is surjective and C_0 is a projective module.

For the induction step assume that $u_0, \ldots, u_i$ have been constructed. This yields a commutative diagram:

$$\begin{tikzcd} C_{i+1} \arrow[r, "\partial_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "u_i"] \arrow[dr, phantom, "\circlearrowleft"] & C_{i-1} \arrow[d, "u_{i-1}"] \\ C_{i+1}' \arrow[r, "\partial_{i+1}'"] & C_i' \arrow[r, "\partial_i'"] & C_{i-1}' \end{tikzcd}$$

where the bottom row is exact (for $i=0$ set $u_{-1} = \varphi: H_0(C_\cdot) \longrightarrow H_0(C_\cdot'))$. Thus $u_i(\operatorname{im}\partial_{i+1}) \subseteq u_i(\ker\partial_i) \subseteq \ker\partial_i' = \operatorname{im}\partial_{i+1}'$. Since C_{i+1} is projective, we may lift the map $u_i\partial_{i+1}$ to an A-linear map $u_{i+1}: C_{i+1} \longrightarrow C_{i+1}'$ with $\partial_{i+1}' u_{i+1} = u_i\partial_{i+1}$.

Uniqueness: We show that if $u_\cdot$ is a morphism of complexes (with the properties of (a)) with $H_0(u_\cdot) = 0$, then $u_\cdot \sim 0$. We will construct the homotopy maps s_i inductively.
For $i=0$, since $H_0(u_\cdot) = 0$ we have $\operatorname{im} u_0 \subseteq \operatorname{im}\partial_1'$:

$$\begin{tikzcd} & C_0 \arrow[d, "u_0"] \\ C_1' \arrow[r, "\partial_1'"] & C_0' \end{tikzcd}$$

Since C_0 is projective, u_0 can be lifted to $s_0: C_0 \longrightarrow C_1'$ with $u_0 = \partial_1' s_0$. For the induction step assume that $s_0, \ldots, s_i$ have been constructed.

$$\begin{tikzcd} C_{i+2} \arrow[r, "\partial_{i+2}"] & C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "u_{i+1}"'] & C_i \arrow[r, "\partial_i"] \arrow[dl, "s_i"'] \arrow[d, "u_i"] & C_{i-1} \arrow[dl, "s_{i-1}"'] \arrow[d, "u_{i-1}"] \\ C_{i+2}' \arrow[r, "\partial_{i+2}'"] & C_{i+1}' \arrow[r, "\partial_{i+1}'"] & C_i' \arrow[r, "\partial_i'"] & C_{i-1}' \end{tikzcd}$$

Then $\partial_{i+1}'(u_{i+1} - s_i\partial_{i+1}) = \partial_{i+1}' u_{i+1} - (\partial_{i+1}' s_i)\partial_{i+1} = \partial_{i+1}' u_{i+1} - (u_i - s_{i-1}\partial_i)\partial_{i+1} = \partial_{i+1}' u_i - u_i\partial_{i+1} = 0$.
Thus $\operatorname{im}(u_{i+1} - s_i\partial_{i+1}) \subseteq \ker\partial_{i+1}' = \operatorname{im}\partial_{i+2}'$, where the last equality follows since $i+1 > 0$ and $C_\cdot'$ is acyclic. Since C_{i+1} is projective there exists an A-linear map $s_{i+1}: C_{i+1} \longrightarrow C_{i+2}'$ so that $u_{i+1} - s_i\partial_{i+1} = \partial_{i+2}' s_{i+1}$.

(b) follows by similar arguments.

(7.13) Corollary: Let C and C' be projective (injective) resolutions of a module M. Then there exist morphisms of complexes $u: C \longrightarrow C'$ and $v: C' \longrightarrow C$ with $uv \sim id$ and $vu \sim id$.

(7.14) Definition: Let M be an A-module.
(a) If M has a finite projective resolution $0 \to P_n \longrightarrow \dots \longrightarrow P_0 \to 0$, then M is said to have finite projective dimension. In this case the smallest possible n is called the projective dimension of M. Notation: $\operatorname{projdim}_A M = \operatorname{projdim} M$.
(b) If M has a finite injective resolution $0 \to I_0 \longrightarrow \dots \longrightarrow I_n \longrightarrow 0$, then M is said to have finite injective dimension. In this case the smallest possible n is called the injective dimension of M. Notation: $\operatorname{injdim}_A M = \operatorname{injdim} M$.

(7.15) Definition: (a) A free resolution of a module M is a projective resolution $F_\bullet$ of M with F_i free for all i.
(b) Let (A, m) be a Noetherian local ring and M a finitely generated A-module. A minimal free resolution of M is a free resolution $(F_\bullet, \partial_\bullet)$ of M with F_i a finite A-module for all i and $\operatorname{im} \partial_{i+1} \subseteq m F_i$ for every i.

(7.16) Remark: Let (A, m) be a Noetherian local ring and M a finitely generated A-module. By (6.68) M has a minimal free resolution.

(7.17) Proposition: Let (A, m) be a Noetherian local ring, M a finite A-module and $F_\bullet$ a minimal free resolution of M. Then
(a) $F_\bullet$ is unique up to isomorphism
(b) If $P_\bullet$ is a projective resolution of M, then $F_\bullet$ is isomorphic to a direct summand of $P_\bullet$.

Proof: (a) Follows from (b).

(b) By (7.13) there are morphisms of complexes $v_\bullet : F_\bullet \longrightarrow P_\bullet$ and $w_\bullet : P_\bullet \longrightarrow F_\bullet$ so that $w_\bullet v_\bullet \sim id_{F_\bullet}$. Write $u_\bullet = w_\bullet v_\bullet$. We claim that $u_\bullet$ is an isomorphism. Since $u_\bullet \sim id_{F_\bullet}$, for all i : $u_i = id_{F_i} + \partial_{i+1} s_i + s_{i-1}\partial_i$. Since $F_\bullet$ is a minimal resolution, $im(\partial_{i+1} s_i + s_{i-1}\partial_i) \subseteq m F_i$. Thus $F_i = im\, u_i + m F_i$ and by Nakayama's lemma $F_i = im\, u_i$. Thus $u_i : F_i \longrightarrow F_i$ is surjective and hence an isomorphism. This implies that $w_i : P_i \longrightarrow F_i$ is surjective for all i and that the complex $F_\bullet$ is a direct summand of the complex $P_\bullet$.

Let M be an A-module. In the following we denote by $E_A(M)$ or $E(M)$ the injective hull (envelope) of M (6.89).

(7.18) Definition: A minimal injective resolution of a module M is an injective resolution $(E^\bullet, \partial^\bullet)$ if $E^0 = E(M)$ and $E^{i+1} = E(\mathrm{coker}\, \partial^{i-1})$.

(7.19) Remark: Let M be an A-module. Then M has a minimal injective resolution.

Similar to (7.17) one can show:

(7.20) Proposition: Let M be an A-module and $E^\bullet$ a minimal injective resolution of M.

(a) $E^\bullet$ is unique up to isomorphism.

(b) If $I^\bullet$ is an injective resolution of M then $E^\bullet$ is isomorphic to a direct summand of $I^\bullet$.

§2: DERIVED FUNCTORS

(7.21) Definition: a functor (contravariant functor) $F: A\text{-mod} \to B\text{-mod}$ is additive if for any two A-modules M and M', the induced map $\mathrm{Hom}_A(M,M') \to \mathrm{Hom}_B(F(M),F(M'))$ ($\mathrm{Hom}_A(M,M') \to \mathrm{Hom}_B(F(M'),F(M))$, respectively) is a homomorphism of abelian groups.

(7.22) Examples: Let N be an A-module and $I \subseteq A$ an ideal.

(a) $F = - \otimes_A N : A\text{-mod} \to A\text{-mod}$ given by $F(M) = M \otimes_A N$ and $F(f) = f \otimes_A \mathrm{id}_N$ is an additive functor which is right exact. F is exact if and only if N is flat.

(b) $F = \mathrm{Hom}_A(N,-): A\text{-mod} \to A\text{-mod}$ given by $F(M) = \mathrm{Hom}_A(N,M)$ and $F(f) = \mathrm{Hom}_A(N,f)$ is an additive functor which is left exact. It is exact if and only if N is projective.

(c) $F = \mathrm{Hom}_A(-,N): A\text{-mod} \to A\text{-mod}$ given by $F(M) = \mathrm{Hom}_A(M,N)$ and $F(f) = \mathrm{Hom}_A(f,N)$ is an additive contravariant functor which is left exact. It is exact if and only if N is injective.

(d) $F = \Gamma_I : A\text{-mod} \to A\text{-mod}$ given by $F(M) = \Gamma_I(M)$ and $F(f) = \Gamma_I(f)$ is an additive functor which is left exact.

Let $F: A\text{-mod} \to B\text{-mod}$ be an additive functor. For a complex $(C_\cdot, \partial_\cdot)$ of A-modules let $F(C_\cdot)$ be the complex of B-modules with $F(C_\cdot)_i = F(C_i)$ and $\partial_i^{F(C_\cdot)} = F(\partial_i)$. Since F is additive, $(F(C_\cdot), \partial_\cdot^{F(C_\cdot)})$ is a complex. For a morphism of complexes $u_\cdot : C_\cdot \to C'_\cdot$ let $F(u_\cdot): F(C_\cdot) \to F(C'_\cdot)$ be given by $F(u)_i = F(u_i)$. This is a morphism of complexes. Let $u_\cdot, v_\cdot$ be morphisms of complexes so that $u_\cdot \sim v_\cdot$, then $F(u_\cdot) \sim F(v_\cdot)$ since F is additive. In particular, if the complex $C_\cdot$ has a contracting homotopy, then so does $F(C_\cdot)$.

For every A-module M fix a projective resolution $P_{M\cdot}$. Define $L_i F(M) = H_i(F(P_{M\cdot}))$. Let $\varphi: M \to M'$ be an A-linear map. By (7.12) there is a morphism of complexes

$u_\cdot : P_{M\cdot} \longrightarrow P_{M'\cdot}$ with $H_0(u_\cdot) = \varphi$. Define $L_iF(\varphi) : L_iF(M) \longrightarrow L_iF(M')$ by $L_iF(\varphi) = H_i(F(u_\cdot))$. This is well defined since if $v_\cdot$ is another morphism of complexes with $H_0(v_\cdot) = \varphi$, then by (7.13) $u_\cdot \sim v_\cdot$. Thus $F(u_\cdot) \sim F(v_\cdot)$ and by (7.8) $H_i(F(u_\cdot)) = H_i(F(v_\cdot))$. One easily checks that L_iF: A-mod $\longrightarrow$ B-mod are additive functors.

(7.23) <u>Definition</u>: The functors L_iF are called <u>left</u> <u>derived</u> <u>functors</u> of F.

(7.24) <u>Definition</u>: Two functors F, G: A-mod $\longrightarrow$ B-mod are <u>naturally</u> <u>equivalent</u>, $F \simeq G$, if for every A-module M there is an isomorphism $t_M : F(M) \xrightarrow{\sim} G(M)$ so that for every $f \in \mathrm{Hom}_A(M, M')$ the following diagram commutes:

$$\begin{array}{ccc} F(M) & \xrightarrow[\sim]{t_M} & G(M) \\ F(f) \downarrow & & \downarrow G(f) \\ F(M') & \xrightarrow[\sim]{t_{M'}} & G(M') \end{array}$$

(similarly for contravariant functors)

For every A-module M fix some other projective resolution $\hat{P}_M$ and use these to define $\hat{L}_iF$.

(7.25) <u>Proposition</u>: $L_iF \simeq \hat{L}_iF$

<u>Proof</u>: For every A-module M by (7.13) there are morphisms of complexes $u_\cdot : P_{M\cdot} \longrightarrow \hat{P}_{M\cdot}$ and $v_\cdot : \hat{P}_{M\cdot} \longrightarrow P_{M\cdot}$ with $u_\cdot v_\cdot \sim \mathrm{id}_{\hat{P}_M}$ and $v_\cdot u_\cdot \sim \mathrm{id}_{P_M}$. Hence $F(u_\cdot) \cdot F(v_\cdot) \sim \mathrm{id}_{F(\hat{P}_M)}$ and $F(v_\cdot) F(u_\cdot) \sim \mathrm{id}_{F(P_M)}$. By (7.8) $H_i(F(u_\cdot)) H_i(F(v_\cdot)) = \mathrm{id}$ and $H_i(F(v_\cdot)) H_i(F(u_\cdot)) = \mathrm{id}$ and $t_M = H_i(F(u_\cdot)) : L_iF(M) \xrightarrow{\sim} \hat{L}_iF(M)$ is an isomorphism of B-modules.
Let $\varphi : M \longrightarrow M'$ be an A-linear map. Using (7.12) and (7.8) again one shows that the diagram:

$$\begin{array}{ccc} L_iF(M) & \xrightarrow{t_M} & \hat{L}_iF(M) \\ L_iF(\varphi) \downarrow & & \downarrow \hat{L}_iF(\varphi) \\ L_iF(M') & \xrightarrow{t_{M'}} & \hat{L}_iF(M') \end{array}$$

commutes.

(7.26) <u>Proposition</u>: (a) If P is a projective module then $L_iF(P) = 0$ for all $i > 0$.

(b) If M has finite projective dimension then $L_iF(M)=0$ for all $i>\operatorname{projdim} M$.

(c) If $0\longrightarrow K_n\longrightarrow P_{n-1}\longrightarrow\dots\longrightarrow P_0\longrightarrow M\longrightarrow 0$ is an exact sequence with P_j projective (K_n is called an <u>n-th syzygy module</u>) then $L_iF(M)\cong L_{i-n}F(K_n)$ for all $i>n$.

(d) If F is exact then $L_iF=0$ for all $i>0$.

(e) If F is right exact then $L_0F\simeq F$.

<u>Proof</u>: (c) Let $\dots P_{n+1}\longrightarrow P_n\longrightarrow 0$ be a projective resolution of K_n. Then $\dots\longrightarrow P_{n+1}\longrightarrow P_n\longrightarrow P_{n-1}\longrightarrow\dots\longrightarrow P_0\longrightarrow 0$ is a projective resolution of M. The statement follows from the definition of L_iF.

(d) Let $P_\cdot:\ \dots\longrightarrow P_1\longrightarrow P_0\longrightarrow 0$ be a projective resolution of a module M. Then $P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$ is exact. Since F is right exact, $F(P_1)\longrightarrow F(P_0)\longrightarrow F(M)\longrightarrow 0$ is exact. Thus $F(M)\cong H_0(F(P_\cdot))=L_0F(M)$. It is easy to see that this isomorphism is natural.

(7.27) <u>Lemma</u>: (Horseshoe Lemma) Let $0\longrightarrow M'\xrightarrow{\varphi}M\xrightarrow{\psi}M''\longrightarrow 0$ be an exact sequence of A-modules and let $P_\cdot'$ and $P_\cdot''$ be projective resolutions of M and M'. Then there exists an exact sequence of morphisms of complexes $0\longrightarrow P_\cdot'\xrightarrow{u_\cdot}P_\cdot\xrightarrow{v_\cdot}P_\cdot''\longrightarrow 0$ so that $P_\cdot$ is a projective resolution of M and $H_0(u_\cdot)=\varphi$ and $H_0(v_\cdot)=\psi$.

<u>Proof</u>: Consider the diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_1' \arrow[d] & & K_1'' \arrow[d] & \\
 & P_0' \arrow[d, "\pi'"'] & & P_0'' \arrow[d, "\pi''"] & \\
0 \arrow[r] & M' \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & M'' \arrow[r] & 0
\end{tikzcd}$$

Define $P_0 = P_0' \oplus P_0''$ and let $\pi : P_0 \longrightarrow M$ be the A-linear map with $\pi|_{P_0'} = \varphi\pi'$ and $\pi|_{P_0''}$ any lifting of the map π'' (such a lifting exists since P_0'' is projective and ψ is surjective). Let $u_0 : P_0' \longrightarrow P_0$ and $v_0 : P_0 \longrightarrow P_0''$ be the canonical maps. The diagram:

$$\begin{tikzcd} 0 \arrow[r] & P_0' \arrow[r, "u_0"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r, "v_0"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowleft"] & P_0'' \arrow[r] \arrow[d, "\pi''"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & M'' \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows. By the Snake Lemma (7.5) π is surjective and u_0 and v_0 induce an exact sequence $0 \longrightarrow K_1' \xrightarrow{\varphi_1} K_1 = \ker(\pi) \xrightarrow{\psi_1} K_1'' \longrightarrow 0$. Continue.

(7.28) <u>Theorem</u>: Let $0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of A-modules. Then there is a long exact sequence

$$\cdots\ L_iF(M') \xrightarrow{L_iF(\varphi)} L_iF(M) \xrightarrow{L_iF(\psi)} L_iF(M'') \xrightarrow{\Delta_i} L_{i-1}F(M') \longrightarrow \cdots$$
$$\cdots \longrightarrow L_0F(M') \longrightarrow L_0F(M) \longrightarrow L_0F(M'') \longrightarrow 0$$

<u>Proof</u>: By (7.27) there is an exact sequence of morphisms of complexes

$$0 \longrightarrow P_\cdot' \xrightarrow{u_\cdot} P_\cdot \xrightarrow{v_\cdot} P_\cdot'' \longrightarrow 0$$

where $P_\cdot', P_\cdot, P_\cdot''$ are projective resolutions of M', M, M'', and $H_0(u_\cdot) = \varphi$, $H_0(v_\cdot) = \psi$. For all i the sequence $0 \to P_i' \xrightarrow{u_i} P_i \xrightarrow{v_i} P_i'' \longrightarrow 0$ is split exact since P_i'' is projective. Hence $0 \to F(P_i') \xrightarrow{F(u_i)} F(P_i) \xrightarrow{F(v_i)} F(P_i'') \longrightarrow 0$ is exact and

$$0 \longrightarrow F(P_\cdot') \xrightarrow{F(u_\cdot)} F(P_\cdot) \xrightarrow{F(v_\cdot)} F(P_\cdot'') \longrightarrow 0$$

is an exact sequence of morphisms of complexes. By (7.6) there is a long exact sequence of homology:

$$\cdots \longrightarrow H_i(F(P_\cdot')) \xrightarrow{H_i(F(u_\cdot))} H_i(F(P_\cdot)) \xrightarrow{H_i(F(v_\cdot))} H_i(F(P_\cdot'')) \xrightarrow{\Delta_i} H_{i-1}(F(P_\cdot')) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(F(P_\cdot'')) \longrightarrow H_{-1}(F(P_\cdot'')) = 0$$

The assertion follows from the definition of the functors L_iF.

(7.29) <u>Remark</u>: Let F be right exact. Then every exact sequence $0 \to M' \to M \to M'' \to 0$

induces a long exact sequence:

$$\cdots \longrightarrow L_1F(M') \longrightarrow L_1F(M) \longrightarrow L_1F(M'') \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0.$$

(7.30) <u>Theorem</u>: Let

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\varphi"] \arrow[d, "\gamma'"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r, "\psi"] \arrow[d, "\gamma"] \arrow[dr, phantom, "\circlearrowleft"] & M'' \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of A-linear maps with exact rows. Then

$$\begin{tikzcd}
\cdots \arrow[r] & L_iF(M') \arrow[r] \arrow[d, "L_iF(\gamma')"] \arrow[dr, phantom, "\circlearrowleft"] & L_iF(M) \arrow[r] \arrow[d, "L_iF(\gamma)"] \arrow[dr, phantom, "\circlearrowleft"] & L_iF(M'') \arrow[r, "\Delta_i"] \arrow[d, "L_iF(\gamma'')"] \arrow[dr, phantom, "\circlearrowleft"] & L_{i-1}F(M') \arrow[r] \arrow[d, "L_{i-1}F(\gamma')"] & \cdots \\
\cdots \arrow[r] & L_iF(N') \arrow[r] & L_iF(N) \arrow[r] & L_iF(N'') \arrow[r, "\widetilde{\Delta}_i"] & L_{i-1}F(N') \arrow[r] & \cdots
\end{tikzcd}$$

is a commutative diagram with exact rows.

<u>Proof</u>: The result follows from (7.6)(c), the naturality of the longexact sequence of homology, once we have shown the following:

(7.31) <u>Lemma</u>: Let

$$\begin{tikzcd}
0 \arrow[r] & P_\cdot' \arrow[r, "u_\cdot"] \arrow[d, "g_\cdot'"'] & P_\cdot \arrow[r, "v_\cdot"] \arrow[d, dotted] & P_\cdot'' \arrow[r] \arrow[d, "g_\cdot''"] & 0 \\
0 \arrow[r] & Q_\cdot' \arrow[r, "x_\cdot"] & Q_\cdot \arrow[r, "y_\cdot"] & Q_\cdot'' \arrow[r] & 0
\end{tikzcd}$$

be morphisms of complexes with exact rows so that $P_\cdot', P_\cdot, P_\cdot'', Q_\cdot', Q_\cdot, Q_\cdot''$ are projective resolutions of M', M, M'', N', N, N'' and the morphisms $u_\cdot, v_\cdot, g_\cdot', g_\cdot'', x_\cdot, y_\cdot$ induce the maps $\varphi, \psi, \gamma', \gamma'', \varepsilon, \mu$. Then there exists a morphism of complexes $g_\cdot : P_\cdot \longrightarrow Q_\cdot$ inducing γ so that the above diagram commutes.

<u>Proof</u>: We construct g_i inductively. To define g_0, we may assume $P_0 = P_0' \oplus P_0''$, $Q_0 = Q_0' \oplus Q_0''$ and that u_0, v_0, x_0, y_0 are the natural embeddings and projections.

$$\begin{tikzcd}
 & P_0' \arrow[rr, "u_0"] \arrow[dl, "g_0'"'] \arrow[dd, "\pi'"] & & P_0 \arrow[rr] \arrow[dl, dotted, "g_0"] \arrow[dd, "\pi"] & & P_0'' \arrow[dl, "g_0''"] \arrow[dd, "\pi''"] \\
Q_0' \arrow[rr, "x_0"] \arrow[dd, "\tau'"'] & & Q_0 \arrow[rr] \arrow[dd, "\tau"] & & Q_0'' \arrow[dd, "\tau''"] & \\
 & M' \arrow[rr, "\varphi"] \arrow[dl, "\gamma'"] & & M \arrow[rr, "\psi"] \arrow[dl, "\gamma"] & & M'' \arrow[dl, "\gamma''"] \\
N' \arrow[rr, "\varepsilon"] & & N \arrow[rr, "\mu"] & & N'' &
\end{tikzcd}$$

Write $p = \pi|_{P_0''}$ and $t = \tau|_{Q_0''}$. Define $g_0 : P_0 = P_0' \oplus P_0'' \longrightarrow Q_0 = Q_0' \oplus Q_0''$ by $g_0 = \begin{pmatrix} g_0' & f \\ 0 & g_0'' \end{pmatrix}$ where $f : P_0'' \longrightarrow Q_0'$ is yet to be determined. Note that the two rectangles on the top commute already. We have to determine f so that $\tau g_0 = \gamma\pi$. We have that $\tau g_0 u_0 = \tau x_0 g_0' = \varepsilon\tau' g_0'$ and $\gamma\pi u_0 = \gamma\varphi\pi' = \varepsilon\gamma'\pi' = \varepsilon\tau' g_0'$ and therefore $\tau g_0|_{P_0'} = \gamma\pi|_{P_0'}$. Thus $\tau g_0 = \gamma\pi$ if and only if $\tau g_0|_{P_0''} = \gamma\pi|_{P_0''}$, which means $\varepsilon\tau' f + t g_0'' = \gamma p$, or equivalently, $\varepsilon\tau' f = \gamma p - t g_0''$. (Note that $g_0'' = g_0|_{P_0''}$.) Since P_0'' is projective it follows that such an f exists if $\operatorname{im}(\gamma p - t g_0'') \subseteq \operatorname{im}(\varepsilon\tau')$:

$$\begin{tikzcd}
 & P_0'' \arrow[dl, dotted, "f"'] \arrow[d, "\gamma p - t g_0''"] & \\
Q_0' \arrow[r, "\varepsilon\tau'"] & \operatorname{im}(\varepsilon\tau') \arrow[r] & 0
\end{tikzcd}$$

But $\mu\gamma p = \gamma''\psi p = \gamma''\pi'' = \tau'' g_0'' = \mu t g_0''$ and $\mu(\gamma p - t g_0'') = 0$. Hence $\operatorname{im}(\gamma p - t g_0'') \subseteq \ker\mu = \operatorname{im}(\varepsilon) = \operatorname{im}(\varepsilon\tau')$, where the last equality follows from the surjectivity of τ'. Continue (with the same argument) by replacing M', M, M'', N', N, N'' by $\ker\pi', \ker\pi, \ker\pi'', \ker\tau', \ker\tau, \ker\tau''$ etc.

(7.32) <u>Remark</u>: Theorem (7.30) and its proof also show that the maps Δ_i constructed in the proof of (7.28) are determined by the exact sequence $0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ and do not depend on $0 \longrightarrow P_\cdot' \xrightarrow{u_\cdot} P_\cdot \xrightarrow{v_\cdot} P_\cdot'' \longrightarrow 0$.

For every A-module M fix an injective resolution $I_M^\cdot$. Let $F : A\text{-mod} \longrightarrow B\text{-mod}$ be an additive functor. Define $R^iF(M) = H^i(F(I_M^\cdot))$. Let $\varphi : M \longrightarrow M'$ be an A-linear map. By (7.12) there is a morphism of complexes $u^\cdot : I_M^\cdot \longrightarrow I_{M'}^\cdot$ with $H^0(u^\cdot) = \varphi$. Define

$R^iF(\varphi): R^iF(M) \longrightarrow R^iF(M')$ by $R^iF(\varphi) = H^i(F(u'))$. By (7.12) and (7.8), $R^iF(\varphi)$ is well defined. R^iF: A-mod $\longrightarrow$ B-mod are additive functors. One can show as in (7.25) that they are independent of the choices of injective resolutions.

(7.33) <u>Definition</u>: The functors R^iF are called <u>right</u> <u>derived</u> <u>functors</u> of F.

(7.34) <u>Theorem</u>: (a) If E is an injective module then $R^iF(E)=0$ whenever $i>0$.
(b) If M has finite injective dimension then $R^iF(M)=0$ for all $i > \operatorname{inj\,dim} M$.
(c) If $0 \to M \to I^0 \to \dots \to I^{n-1} \to L^n \to 0$ is exact with I^j injective then $R^iF(M) = R^{i-n}F(L^n)$ for all $i>n$.
(d) If F is left exact then $R^0F \simeq F$.

(7.35) <u>Theorem</u>: (a) Let $0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of A-modules. Then there is a long exact sequence:

$$0 \to R^0F(M') \to R^0F(M) \to R^0F(M'') \to \dots$$
$$\dots \to R^{i-1}F(M'') \xrightarrow{\Delta^i} R^iF(M') \xrightarrow{R^iF(\varphi)} R^iF(M) \xrightarrow{R^iF(\psi)} R^iF(M'') \to \dots$$

(b) The long exact sequence of (a) is natural.

Let F: A-mod $\longrightarrow$ B-mod be an additive contravariant functor. For every A-module M fix a projective resolution $P_M.$ and an injective resolution $I^{\cdot}_M$. Define $R^iF(M) = H^i(F(P_M.))$, $L_iF(M) = H_i(F(I^{\cdot}_M))$, and for an A-linear map $\varphi: M \longrightarrow M'$ define $R^iF(\varphi): R^iF(M') \to R^iF(M)$ and $L_iF(\varphi): L_iF(M') \longrightarrow L_iF(M)$ in the obvious way. R^iF and L_iF are additive contravariant functors whose definitions do not depend on the choices of projective, injective resolutions. The functors R^iF are called <u>right</u> <u>derived</u> <u>functors</u> and L_iF <u>left</u> <u>derived</u> <u>functors</u> of F.

(7.36) <u>Theorem</u>: Let F be an additive contravariant functor.
(a) If P is a projective module then $R^iF(P)=0$ for all $i>0$.
(b) If M has a finite projective dimension then $R^iF(M)=0$ for all $i > \operatorname{proj\,dim} M$.

(c) If K_n is an nth syzygy module of M then $R^iF(M) = R^{i-n}F(K_n)$ for all $i>n$.

(d) If F is left exact then $R^0F \simeq F$.

(e) If $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is an exact sequence then there is a long exact sequence:

$$0 \to R^0F(M'') \to R^0F(M) \to R^0F(M') \to \cdots$$
$$\cdots \to R^{i-1}F(M') \xrightarrow{\Delta^i} R^iF(M'') \xrightarrow{R^iF(\psi)} R^iF(M) \xrightarrow{R^iF(\varphi)} R^iF(M') \to \cdots$$

This long exact sequence is natural.

§3: TOR AND EXT

(7.37) Definition: Let N be an A-module.

(a) $\mathrm{Tor}_i^A(-,N) = L_i(-\otimes_A N)$

(b) $\mathrm{Tor}_i^A(N,-) = L_i(N\otimes_A -)$

(c) $\mathrm{Ext}_A^i(-,N) = R^i\mathrm{Hom}_A(-,N)$

(d) $\mathrm{Ext}^i_A(N,-) = R^i\,\mathrm{Hom}_A(N,-)$

$\mathrm{Tor}_i^A(-,N)$ and $\mathrm{Tor}_i^A(N,-)$ are additive functors. Since $-\otimes_A N \simeq N\otimes_A -$, $\mathrm{Tor}_i^A(-,N) \simeq \mathrm{Tor}_i^A(N,-)$. $\mathrm{Tor}_0^A(-,N) \simeq -\otimes_A N$, since $-\otimes_A N$ is rightexact. Moreover, $\mathrm{Tor}_i^A(-,N)(P)=0$ if P is projective and $i>0$.

(7.38) Theorem: If $0\longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ is an exact sequence, then there is a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_i(M',N) \xrightarrow{\mathrm{Tor}_i(\varphi,N)} \mathrm{Tor}_i(M,N) \xrightarrow{\mathrm{Tor}_i(\psi,N)} \mathrm{Tor}_i(M'',N) \xrightarrow{\Delta_i} \mathrm{Tor}_{i-1}(M',N) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{Tor}_1(M'',N) \longrightarrow M'\otimes N \longrightarrow M\otimes N \longrightarrow M''\otimes N \longrightarrow 0$$

Furthermore this sequence is natural.

$\mathrm{Ext}_A^i(N,-)$ is an additive functor, $\mathrm{Ext}_A^0(N,-) \simeq \mathrm{Hom}_A(N,-)$ (since $\mathrm{Hom}_A(N,-)$ is left exact), and $\mathrm{Ext}_A^i(N,-)(I)=0$ if I is injective and $i>0$.
$\mathrm{Ext}_A^i(-,N)$ is an additive contravariant functor, $\mathrm{Ext}_A^0(-,N) \simeq \mathrm{Hom}_A(-,N)$ (since $\mathrm{Hom}_A(-,N)$ is left exact), and $\mathrm{Ext}_R^i(-,N)(P)=0$ if P is projective and $i>0$.

(7.39) Theorem: If $0\longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence, then there are long exact sequences:

(a) $0 \longrightarrow \mathrm{Hom}(N,M') \longrightarrow \mathrm{Hom}(N,M) \longrightarrow \mathrm{Hom}(N,M'') \longrightarrow \mathrm{Ext}^1(N,M') \longrightarrow \cdots$

(b) $0 \longrightarrow \mathrm{Hom}(M'',N) \longrightarrow \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(M',N) \longrightarrow \mathrm{Ext}^1(M'',N) \longrightarrow \cdots$

Furthermore these sequences are natural.

The symbols $\mathrm{Tor}_i^A(M,N)$ and $\mathrm{Ext}_A^i(M,N)$ are well defined due to the following fact:

(7.40) <u>Theorem</u>: (a) $\mathrm{Tor}_i^A(-,N)(M) \cong \mathrm{Tor}_i^A(M,-)(N)$

(b) $\mathrm{Ext}_A^i(-,N)(M) \cong \mathrm{Ext}_A^i(M,-)(N)$

<u>Proof</u>: We only prove (b), the proof of (a) is similar.

Consider a projective resolution $P.$ of M and an injective resolution $I^{\cdot}$ of N and write

$$\begin{tikzcd}[column sep=small]
\cdots & P_2 \arrow[rr] \arrow[dr] && P_1 \arrow[rr] \arrow[dr] && P_0 \arrow[r] & M \arrow[r] & 0 \\
&& K_2 \arrow[ur] \arrow[dr] && K_1 \arrow[ur] \arrow[dr] &&& \\
& 0 \arrow[ur] && 0 \arrow[ur] && 0 &&
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
0 \arrow[r] & N \arrow[r] & I^0 \arrow[rr] \arrow[dr] && I^1 \arrow[rr] \arrow[dr] && I^2 \arrow[r] & \cdots \\
&&& L^1 \arrow[ur] \arrow[dr] && L^2 \arrow[ur] \arrow[dr] && \\
&& 0 \arrow[ur] && 0 \arrow[ur] && 0 &
\end{tikzcd}$$

The sequences $0 \to K_1 \to P_0 \to M \to 0$ and $0 \to N \to I^0 \to L^1 \to 0$ together with (7.39) yield a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] && \\
0 \arrow[r] & \mathrm{Hom}(M,N) \arrow[r] \arrow[d] & \mathrm{Hom}(M,I^0) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}(M,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1(M,-)(N) \arrow[r] & \mathrm{Ext}^1(M,-)(I^0)=0 \\
0 \arrow[r] & \mathrm{Hom}(P_0,N) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,I^0) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,L^1) \arrow[r] \arrow[d, "\gamma"] & 0 & \\
0 \arrow[r] & \mathrm{Hom}(K,N) \arrow[r] \arrow[d] & \mathrm{Hom}(K,I^0) \arrow[r, "\beta"] \arrow[d] & \mathrm{Hom}(K,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1(K,-)(N) \arrow[r] & 0 \\
& \mathrm{Ext}^1(-,N)(M) \arrow[d] & 0 & \mathrm{Ext}^1(-,L^1)(M) \arrow[d] && \\
& \mathrm{Ext}^1(-,N)(P_0)=0 && 0 &&
\end{tikzcd}$$

By the Snake Lemma (7.5), $\mathrm{coker}\,\alpha \cong \mathrm{Ext}^1(-,N)(M)$, whereas by the first row of the diagram $\mathrm{coker}\,\alpha \cong \mathrm{Ext}^1(M,-)(N)$. Thus $\mathrm{Ext}^1(-,N)(M) \cong \mathrm{Ext}^1(M,-)(N)$ and we may write $\mathrm{Ext}^1(M,N)$ for this module.

In the above diagram we also have $\mathrm{im}\,\beta = \mathrm{im}\,\gamma$ and therefore $\mathrm{coker}\,\gamma \cong \mathrm{coker}\,\beta$ which gives:

$$(*) \quad \mathrm{Ext}^1(M,L^1) \cong \mathrm{Ext}^1(K_1,N)$$

Let $i>1$. Then $\quad \mathrm{Ext}^i(-,N)(M) \cong \mathrm{Ext}^1(-,N)(K_{i-1})$ by (7.36)(c)

$$\begin{aligned}
&\cong \mathrm{Ext}^1(K_{i-1},N) \\
&\cong \mathrm{Ext}^1(K_{i-2},L^1) \cong \dots \cong \mathrm{Ext}^1(M,L^{i-1}) && \text{by } (*) \\
&\cong \mathrm{Ext}^1(M,-)(L^{i-1}) \\
&\cong \mathrm{Ext}^i(M,-)(N) && \text{by } (7.34)(c).
\end{aligned}$$

(7.41) Proposition: Let M be an A-module and $n>0$ a positive integer. The following are equivalent:

(a) $\mathrm{projdim}\, M \leq n$

(b) Every n-th syzygy module of M is projective.

(c) $\mathrm{Ext}^i_A(M,N)=0$ for all $i>n$ and every A-module N.

(d) $\mathrm{Ext}^{n+1}_A(M,N)=0$ for every A-module N.

Proof: (b) $\Rightarrow$ (a): clear

(a) $\Rightarrow$ (c): (7.36)(b)

(c) $\Rightarrow$ (d): clear

(d) $\Rightarrow$ (b): Let K_n be an n-th syzygy of M. By (7.36)(c), $\mathrm{Ext}^1_A(K_n,N) \cong \mathrm{Ext}^{n+1}_A(M,N)=0$. Since $\mathrm{Ext}^1_A(K_n,N)=0$ for every A-module N, the long exact sequence (7.39)(a) shows that the functor $\mathrm{Hom}_A(K_n,-)$ is exact. Thus K_n is projective.

(7.42) Proposition: Let M be an A-module and $n>0$ a positive integer. The following are equivalent:

(a) $\mathrm{injdim}\, M \leq n$

(b) If $0 \to M \to I^0 \to \dots \to I^{n-1} \to L^n \to 0$ is an exact sequence with I^j injective then L^n is injective.

(c) $\mathrm{Ext}^i_A(N,M)=0$ for all $i>n$ and every A-module N.

(d) $\mathrm{Ext}^{n+1}_A(N,M)=0$ for every A-module N.

(e) $\mathrm{Ext}^{n+1}_A(A/I,M)=0$ for every A-ideal I.

Proof: (e) $\Rightarrow$ (b): By (7.34) $\mathrm{Ext}^1_A(A/I,L^n) \cong \mathrm{Ext}^{n+1}_A(A/I,M)=0$ for every ideal I.

Thus by (7.39)(b), the sequence $\mathrm{Hom}_A(A, L^n) \longrightarrow \mathrm{Hom}_A(I, L^n) \longrightarrow \mathrm{Ext}^1_A(A/I, L^n) = 0$ is exact for every ideal $I \subseteq A$. By (6.27) L^n is injective.

(7.43) Corollary: Let A be a ring, then
$\sup\{\mathrm{projdim}\, M \mid M \text{ an } A\text{-module}\} = \sup\{\mathrm{projdim}\, A/I \mid I \text{ an } A\text{-ideal}\} =$
$\sup\{\mathrm{injdim}\, M \mid M \text{ an } A\text{-module}\} = \sup\{n \mid \mathrm{Ext}^n_A(M,N) \neq 0 \text{ for some } A\text{-modules } M, N\}$.
This (not necessarily finite) number is called the global dimension of A, denoted $\mathrm{gldim}\, A$.

(7.44) Examples: (a) If A is a field then $\mathrm{gldim}\, A = 0$.
(b) If A is a Dedekind domain then $\mathrm{gldim}\, A = 1$ (since every ideal is projective).
(c) $\mathrm{gldim}(\mathbb{Z}/(4)) = \infty$ (Homework)

(7.45) Definition: (a) a flat resolution of a module M is an acyclic complex $F_\bullet$ with flat modules F_i for all i together with an isomorphism $H_0(F_\bullet) \cong M$.
(b) The flat dimension of M, $\mathrm{fldim}_A M = \mathrm{fldim}\, M$, is the minimal length of a flat resolution of M.

(7.46) Proposition: (a) If F is a flat A-module, then $\mathrm{Tor}_i^A(F,N) = 0$ for all $i > 0$ and all A-modules N.
(b) If $F_\bullet$ is a flat resolution of M, then $\mathrm{Tor}_i^A(M,N) \cong H_i(F_\bullet \otimes_A N)$ for all i.

Proof: (a) If F is flat then the functor $F \otimes_A -$ is exact. Thus $\mathrm{Tor}_i^A(F,-) = L_i(F \otimes_A -) = 0$ whenever $i > 0$ by (7.26)(d).
(b) By induction on i: If $i = 0$ then the claim holds since $- \otimes_A N$ is right exact.
Write $0 \to K_1 \to F_0 \to M \to 0$ and $E_\bullet: \dots \to F_2 \to F_1 \to 0$, which is a flat resolution of K_1. Let $i = 1$. By the long exact sequence (7.38), one has an exact sequence $\mathrm{Tor}_1^A(F_0,N) = 0 \to \mathrm{Tor}_1^A(M,N) \to K_1 \otimes_A N \to F_0 \otimes_A N$. Hence
$\mathrm{Tor}_1^A(M,N) \cong \ker(K_1 \otimes N \to F_0 \otimes N) = \ker\left(F_1 \otimes N / \mathrm{im}(F_2 \otimes N) \to F_0 \otimes N\right) = H_1(F_\bullet \otimes_A N)$.

If $i>1$, then by (7.38) $\operatorname{Tor}_i^A(M,N) \cong \operatorname{Tor}_{i-1}^A(K_1,N) \cong H_{i-1}(E_\cdot \otimes_A N) = H_i(F_\cdot \otimes_A N)$.

(7.47) Proposition: The following are equivalent for an integer $n \geq 0$:

(a) $\operatorname{fldim} M \leq n$

(b) If $0 \longrightarrow K_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ is an exact sequence with F_j flat, then K_n is flat.

(c) $\operatorname{Tor}_i^A(M,N) = 0$ for all $i>n$ and every A-module N.

(d) $\operatorname{Tor}_{n+1}^A(M,N) = 0$ for every A-module N.

(e) $\operatorname{Tor}_{n+1}^A(M, A/I) = 0$ for every A-ideal I.

Proof: (a) $\Rightarrow$ (c): follows from (7.46).

(e) $\Rightarrow$ (b): By (7.46): $\operatorname{Tor}_1^A(K_n, A/I) \cong \operatorname{Tor}_{n+1}^A(M, A/I)$ for every A-ideal I. Then $0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$ yields an exact sequence $0 \longrightarrow I \otimes_A K_n \longrightarrow A \otimes_A K_n \cong K_n$. Thus for every ideal I: $I \otimes_A K_n \xrightarrow{\cong} IK_n$ via the natural map. K_n is flat by a Homework problem.

(7.48) Theorem: Let A be a ring, $S \subseteq A$ a multiplicative subset, and M, N A-modules. Then:

(a) $\operatorname{Tor}_i^{S^{-1}A}(S^{-1}M, S^{-1}N) \cong S^{-1}\operatorname{Tor}_i^A(M,N)$

(b) If A is Noetherian and M is finitely generated: $S^{-1}\operatorname{Ext}_A^i(M,N) \cong \operatorname{Ext}^i_{S^{-1}A}(S^{-1}M, S^{-1}N)$.

Proof: (b) By induction on i: If $i=0$ then by (6.62) $S^{-1}\operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$. For $i>0$, consider the exact sequence $0 \longrightarrow K \longrightarrow F_{i-1} \xrightarrow{d} F_i \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ where the F_i are finitely generated free A-modules and $K = \ker d$. With $L = \operatorname{im} d$ we have exact sequences $0 \longrightarrow K \longrightarrow F_{i-1} \longrightarrow L \longrightarrow 0$ and $0 \longrightarrow L \longrightarrow F_{i-2} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$. Since A is Noetherian, the modules K and L are finitely generated. We have a long exact sequence: $0 \longrightarrow \operatorname{Hom}_A(L,N) \longrightarrow \operatorname{Hom}_A(F_{i-1},N) \longrightarrow \operatorname{Hom}_A(K,N) \longrightarrow \operatorname{Ext}_A^1(L,N) \longrightarrow 0$ and therefore $\operatorname{Ext}_A^1(L,N) = \operatorname{coker}(\operatorname{Hom}_A(F_{i-1},N) \longrightarrow \operatorname{Hom}_A(K,N))$. Since localization is exact:
$$S^{-1}(\operatorname{Ext}_A^1(L,N)) = S^{-1}\operatorname{coker}(\operatorname{Hom}_A(F_{i-1},N) \longrightarrow \operatorname{Hom}_A(K,N))$$

$\cong \operatorname{coker}\left(S^{-1}\operatorname{Hom}_A(F_{i-1}, N) \longrightarrow S^{-1}\operatorname{Hom}_A(K,N)\right)$

$\cong \operatorname{coker}\left(\operatorname{Hom}_{S^{-1}A}(S^{-1}F_{i-1}, S^{-1}N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}K, S^{-1}N)\right)$ by (6.62)

Using the exact sequence $0 \to S^{-1}K \to S^{-1}F_{i-1} \to S^{-1}L \to 0$ we see that the last module is isomorphic to $\operatorname{Ext}^1_{S^{-1}A}(S^{-1}L, S^{-1}N)$. Thus $S^{-1}\operatorname{Ext}^1_A(L,N) \cong \operatorname{Ext}^1_{S^{-1}A}(S^{-1}L, S^{-1}N)$.

By (7.26), $\operatorname{Ext}^i_{S^{-1}A}(S^{-1}M, S^{-1}N) \cong \operatorname{Ext}^1_{S^{-1}A}(S^{-1}L, S^{-1}N) \cong S^{-1}\operatorname{Ext}^1_A(L,N) \cong S^{-1}\operatorname{Ext}^i_A(M,N)$.

(a) follows by a similar argument.

(7.49) <u>Corollary</u>: Let A be a Noetherian ring.

(a) $\operatorname{fldim}_A M = \sup\{\operatorname{fldim}_{A_m} M_m \mid m \in m\text{-Spec } A\}$

(b) $\operatorname{projdim}_A M = \sup\{\operatorname{projdim}_{A_m} M_m \mid m \in m\text{-Spec } A\}$ if M is finitely generated.

(c) $\operatorname{injdim}_A M = \sup\{\operatorname{injdim}_{A_m} M_m \mid m \in m\text{-Spec } A\}$

(d) $\operatorname{gldim} A = \sup\{\operatorname{gldim} A_m \mid m \in m\text{-Spec } A\}$

<u>Proof</u>: Use (7.48), (7.47), (7.41), (7.42), (7.43).

§4: MINIMAL RESOLUTIONS

A free resolution of a module M is a projective resolution $F_\cdot$ of M with F_i free for all i. Let (A,m) be a Noetherian local ring and M a finitely generated A-module. A minimal free resolution of M is a free resolution $(F_\cdot, \partial_\cdot)$ of M with F_i finitely generated and $\operatorname{im}\partial_{i+1} \subseteq m F_i$ for all i.

(7.50) Remark and Definition: Let (A,m) be a Noetherian local ring and M a finitely generated A-module. The cardinality of every minimal generating set of M is the same, and is denoted by $\mu(M)$, called the minimal number of generators of M. By Nakayama, $\mu(M) = \dim_{A/m}(M/mM)$.

(7.51) Proposition: Let (A,m) be a Noetherian local ring and M a finitely generated A-module. Then:

(a) M has a minimal free resolution $F_\cdot$

(b) $F_\cdot$ is unique up to isomorphism.

(c) If $P_\cdot$ is a projective resolution of M, then $F_\cdot$ is isomorphic to a direct summand of $P_\cdot$

Proof: (a) Let $b_0 = \mu(M)$ and set $F_0 = A^{b_0}$. Map $F_0 = \bigoplus_{i=1}^{b_0} Ae_i$ onto M. The kernel of this map is contained in mF_0, since otherwise it would contain an element $\sum_{i=1}^{b_0} a_i e_i$ with $a_i \in A - m = A^*$ for some i and $\mu(M) < b_0$, a contradiction. Continue like that.

(b) Follows from (c).

(c) By (7.13), there are morphisms of complexes $v_\cdot : F_\cdot \longrightarrow P_\cdot$, $w_\cdot : P_\cdot \longrightarrow F_\cdot$ so that $w_\cdot v_\cdot \sim \mathrm{id}_{F_\cdot}$. Write $u_\cdot = w_\cdot v_\cdot$. We have to show that $u_\cdot$ is an isomorphism. Since $u_\cdot \sim \mathrm{id}_{F_\cdot}$, for every i: $u_i = \mathrm{id}_{F_i} + \partial_{i+1} s_i + s_{i-1}\partial_i$. Since $F_\cdot$ is a minimal resolution, $\operatorname{im}(\partial_{i+1} s_i + s_{i-1}\partial_i) \subseteq m F_i$ and $F_i = \operatorname{im} u_i + m F_i$. Thus by Nakayama's Lemma $F_i = \operatorname{im} u_i$ and $u_i : F_i \longrightarrow F_i$ is surjective, hence an isomorphism.

(7.52) <u>Definition</u>: Let (A,m) be a Noetherian local ring with $k = A/m$ and M a finitely generated A-module. $b_i(M) = \dim_k \mathrm{Tor}_i^A(k,M)$ is called the i-th <u>Betti number</u> of M.

(7.53) <u>Theorem</u>: Let (A,m) be a Noetherian local ring with $k = A/m$ and M a finitely generated A-module. Then $b_i(M) = \dim_k \mathrm{Ext}_A^i(M,k)$ and for the minimal free A-resolution $F.$ of M one has that $\operatorname{rank} F_i = b_i(M)$.

<u>Proof</u>: Let $(F.,\partial.)$ be the minimal free A-resolution of M. Then $\operatorname{im}\partial_i \subseteq mF_{i-1}$. Thus $k\otimes_A \partial_i = 0$ and $\mathrm{Hom}_A(\partial_i,k) = 0$. Therefore $H.(k\otimes_A F.) = k\otimes_A F.$ and $H^{\cdot}(\mathrm{Hom}_A(F.,k)) = \mathrm{Hom}_A(F.,k)$. Write $F_i = A^{n_i}$. Then $b_i(M) = \dim_k(\mathrm{Tor}_i^A(k,M)) = \dim_k H_i(k\otimes_A F.) = \dim_k k\otimes_A F_i = n_i$ and $\dim_k \mathrm{Ext}_A^i(M,k) = \dim_k H^i(\mathrm{Hom}_A(F.,k)) = \dim_k \mathrm{Hom}_A(F_i,k) = \dim_k \mathrm{Hom}_k(k\otimes_A F_i, k) = n_i$.

(7.54) <u>Corollary</u>: Let A be a Noetherian ring and M a finitely generated A-module. Then $\operatorname{projdim} M = \operatorname{fldim} M$.

<u>Proof</u>: By (7.49) we may assume that A is local with residue field k. Obviously, $\operatorname{projdim} M \geq \operatorname{fldim} M$. By (7.47), $b_i(M) = \mathrm{Tor}_i^A(k,M) = 0$ for $i > \operatorname{fldim} M$. Thus by (7.53) the minimal free resolution of M has length $\leq \operatorname{fldim} M$.

(7.55) <u>Corollary</u>: Let (A,m) be a Noetherian local ring with residue field k. Then $\operatorname{gldim} A = \operatorname{projdim}_A k = \operatorname{fldim}_A k = \operatorname{injdim}_A k$.

<u>Proof</u>: By (7.54) and (7.43) it suffices to prove that for every finitely generated A-module M, $\operatorname{projdim}_A M \leq \operatorname{projdim}_A k$ and $\operatorname{projdim}_A M \leq \operatorname{injdim}_A k$. However, $\mathrm{Tor}_i^A(k,M) = 0$ for $i > \operatorname{projdim}_A k$ and $\mathrm{Ext}_A^i(M,k) = 0$ for $i > \operatorname{injdim}_A k$. Now use (7.53).

Minimal injective resolutions

(7.56) Definition: An A-module M is called indecomposable if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$. Otherwise it is called decomposable.

In the following the injective hull of an A-module M is denoted by $E(M)$ or $E_A(M)$.

(7.57) Remark: Let A be a ring, M an A-module, and $E \subseteq M$ an injective submodule. Then $M = E \oplus F$ for some submodule $F \subseteq M$.

Proof: Consider the diagram
$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, hook, "i"] \arrow[d, "id"] & M \arrow[dl, dashed, "f"] \\ & E & \end{tikzcd}$$
where i is the embedding. Since E is injective, there is an A-linear map $f: M \to E$ with $fi = id_E$. Then $M = E \oplus \ker f$.

(7.58) Proposition: Let A be a Noetherian ring and $P \subseteq A$ a prime ideal.

(a) $E_A(A/p)$ is indecomposable.

(b) Any indecomposable injective A-module is of the form $E_A(A/Q)$ for some $Q \in \operatorname{Spec}(A)$.

Proof: (a) Let $N_1, N_2 \subseteq E(A/p)$ be nonzero submodules. Since $E(A/p)$ is an essential extension of A/p, $N_1 \cap A/p = K_1 \neq 0$ and $N_2 \cap A/p = K_2 \neq 0$. K_1 and K_2 are nonzero ideals of the domain A/p, thus $0 \neq K_1K_2 \subseteq K_1 \cap K_2 \subseteq N_1 \cap N_2$.

(b) Let N be an indecomposable injective A-module. Since A is Noetherian, $\operatorname{Ass}_A(N) \neq \emptyset$. Let $Q \in \operatorname{Ass}_A(N)$, then $A/Q \subseteq N$. Since N is injective there is an A-linear map $\varphi: E(A/Q) \to N$ which extends the embedding $A/Q \hookrightarrow N$. $\ker(\varphi) = (0)$ since $E(A/Q)$ is an essential extension of A/Q and $A/Q \cap \ker(\varphi) = (0)$. $E(A/Q)$ is isomorphic to a submodule of N. By (7.57): $N \cong E(A/Q)$.

(7.59) Proposition: Let A be a Noetherian ring and $P \subseteq A$ a prime ideal.

(a) For every $a \in A-P$ multiplication by a induces an automorphism on $E(A/P)$.
(b) If $Q \in \operatorname{Spec}(A)$ with $P \neq Q$, then $E(A/P) \not\cong E(A/Q)$
(c) For every $\zeta \in E(A/P)$ there is an $n \in \mathbb{N}$ with $P^n \zeta = 0$.

<u>Proof</u>: (a) Let $\varphi: E(A/P) \longrightarrow E(A/P)$ with $\varphi(\zeta) = a\zeta$ be the multiplication by a. Since $\ker(\varphi) \cap A/P = (0)$, it follows that $\ker(\varphi) = (0)$ and therefore $E(A/P) \cong \operatorname{im}(\varphi)$. $\operatorname{im}(\varphi)$ is an injective submodule of $E(A/P)$ with $A/P \subseteq \operatorname{im}(\varphi)$. Thus $E(A/P) = \operatorname{im}(\varphi)$.
(b) If $P \not\subseteq Q$, every element $a \in P-Q$ is a regular element on $E(A/Q)$ but not on $E(A/P)$.
(c) Since $A/P \subseteq E(A/P)$, $\operatorname{Ass}_A(A/P) = \{P\} \subseteq \operatorname{Ass}_A(E(A/P))$. Let $Q \in \operatorname{Ass}_A(E(A/P))$. Then $N = A/Q \subseteq E(A/P)$ and $N \cap A/P \neq (0)$. Therefore $Q \in \operatorname{Ass}_A(A/P)$ and $P = Q$. This shows that $\operatorname{Ass}_A(E(A/P)) = \{P\}$. If $\zeta \in E(A/P)$, then $A\zeta \cong A/\operatorname{ann}(\zeta)$ is a submodule of $E(A/P)$ and thus $\operatorname{Ass}_A(A/\operatorname{ann}(\zeta)) = \{P\}$. Hence $\operatorname{ann}(\zeta)$ is P-primary.

(7.60) <u>Proposition</u>: Let A be a Noetherian ring and $Q \subseteq P \subseteq A$ prime ideals. Then:
(a) $E_A(A/Q)$ is an A_P-module.
(b) $E_A(A/Q) = E_{A_P}(A_P/QA_P)$.

<u>Proof</u>: (a) By (7.59) for every $a \in A-P \subseteq A-Q$ multiplication by a is an isomorphism of $E_A(A/Q)$. Thus $E_A(A/Q)$ is an A_P-module.
(b) By (a): $A/Q \subseteq (A/Q)_P \subseteq E_A(A/Q)$ and (the A_P-module) $E_A(A/Q)$ is an essential extension of the A_P-module $(A/Q)_P$. It remains to show that $E_A(A/Q)$ is injective as an A_P-module. Consider the diagram of A_P-modules and A_P-linear maps:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dashed, "h"] \\ & E(A/Q) & \end{tikzcd}$$

Since f and g are A-linear there is an A-linear map $h: M \longrightarrow E(A/Q)$ with $hf = g$. h is also A_P-linear and $E(A/Q)$ is an injective A_P-module.

(7.61) Example: Let A be a DVR with maximal ideal $m=(p)$, field of quotients $K=Q(A)$, and residue class field $k=A/m$. Then $E_A(A)=K$ and $E_A(k)=K/A$.

Proof: Let $I=(p^r)$ be an ideal of A and $f: I \to K/A$ an A-linear map. We need to extend f to an A-linear map $g: A \to K/A$. Let $f(p^r)=[\alpha]$ for some $\alpha \in K$. Define $g: A \to K/A$ by $g(1)=[\alpha/p^r]$. Thus g extends f and K/A is an injective A-module. Moreover, $k=A/pA \cong p^{-1}A/A \subseteq K/A$. If $\beta \in K$ with $[\beta] \neq 0$ in K/A, then $\beta = u/p^n$ for some $u \in A^*$ and $n>0$. Then $p^{n-1}[\beta]=[p^{n-1}\beta]=[p^{-1}u] \in k$ and K/A is an essential extension of A.

(7.62) Lemma: Let A be a Noetherian ring, $P \in \operatorname{Spec} A$, and M an A-module. Then:

(a) $\operatorname{Ass}_A(E(M)) = \operatorname{Ass}_A(M)$

(b) $\operatorname{Hom}_{A_P}(k(P), E(A/P)_P) \cong k(P)$.

Proof: (a) Since $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(E(M))$, it suffices to show that $\operatorname{Ass}(E(M)) \subseteq \operatorname{Ass}(M)$. Let $Q \in \operatorname{Ass}(E(M))$. Then there exists a submodule $N \subseteq E(M)$ with $N \cong A/Q$. Since $E(M)$ is an essential extension of M, $N \cap M \neq 0$. Thus $\emptyset \neq \operatorname{Ass}(N\cap M) \subseteq \operatorname{Ass}(N)=\{Q\}$. Hence $\{Q\} = \operatorname{Ass}(N \cap M) \subseteq \operatorname{Ass}(M)$.

(b) By (7.60) $E(A/P)_P = E(A/P) = E_{A_P}(k(P))$. Thus we may replace A by A_P to assume that A is local with maximal ideal $P=m$ and residue field $k=A/m$. We have to show that $\operatorname{Hom}_A(k, E(k)) \cong k$. $\operatorname{Hom}_A(k, E(k))$ can be identified with $0 :_{E(k)} m \subseteq E(k)$. Obviously, $k \subseteq 0 :_{E(k)} m$. Suppose $k \subsetneq 0 :_{E(k)} m$. Then the k-vector space $0 :_{E(k)} m$ contains a nontrivial subspace N with $N \cap k = 0$. But this is impossible, since $k \subseteq E(k)$ is an essential extension.

(7.63) Theorem: Let A be a Noetherian ring and E an injective A-module. Then:

(a) E is a direct sum of indecomposable injective A-modules.

(b) For $P \in \operatorname{Spec}(A)$, $E(A/P)$ appears in this decomposition if and only if $P \in \operatorname{Ass}(E)$. The multiplicity with which $E(A/P)$ appears is $\dim_{k(P)} \operatorname{Hom}_{A_P}(k(P), E_P)$. In particular, the direct sum decomposition of E is unique.

Proof: (a) Let $\Gamma = \{S \mid S$ a set of indecomposable injective submodules of E with $\sum_{I\in S} I = \bigoplus_{I\in S} I\}$ be partially ordered by inclusion. If $P \in \mathrm{Ass}(E)$, then $E(A/P) \subseteq E$ and $\Gamma \neq \emptyset$. By Zorn's Lemma Γ has a maximal element S. Set $E' = \bigoplus_{I\in S} I$. Since A is Noetherian, E' is injective (Homework). Thus $E = E' \oplus E''$ by (7.57). If $E'' = 0$, we are done. If $E'' \neq 0$, there exists $P \in \mathrm{Ass}(E'')$ and $E(A/P) \subseteq E''$ since E'' is injective (6.24). Thus $E' \cap E(A/P) = 0$. By (7.58) $E(A/P)$ is an indecomposable injective submodule of E and $S \subsetneq S \cup \{E(A/P)\} \in \Gamma$, contradicting the maximality of S.

(b) Let $E = \bigoplus_{I\in S} I$, where $I \neq 0$ are indecomposable injective submodules of E. Then each I is of the form $E(A/P)$ for some $P \in \mathrm{Spec}(A)$ and $\mathrm{Ass}(E(A/P)) = \{P\}$ (7.62). Finally, $\mathrm{Ass}(E) = \bigcup_{I\in S} \mathrm{Ass}(I)$. This shows the first claim.

In order to show the second claim let $P \in \mathrm{Ass}(E)$. Then

$$\mathrm{Hom}_{A_P}(k(P), E_P) \cong \mathrm{Hom}_{A_P}(k(P), \oplus_{I\in S} I_P) \cong \oplus_{I\in S} \mathrm{Hom}_{A_P}(k(P), I_P)$$

since $k(P)$ is a finitely generated A_P-module. (Homework). By (7.62) $k(P) \cong \mathrm{Hom}_{A_P}(k(P), E(A/P)_P)$. It remains to show that $\mathrm{Hom}_{A_P}(k(P), E(A/Q)_P) = 0$ for $P \neq Q \in \mathrm{Spec}(A)$. If $Q \not\subseteq P$, then $Q \cap (A-P) \neq \emptyset$ and $E(A/Q)_P = 0$ by (7.59)(c). If $Q \subsetneq P$ by (7.59)(a) every element $a \in P-Q$ is a NZD on $E(A/Q)$. Thus no nonzero element of $E(A/Q) = E(A/Q)_P$ is annihilated by P. Thus if $P \neq Q$, $\mathrm{Hom}_{A_P}(k(P), E(A/Q)_P) = 0$.

(7.64) Definition: A minimal injective resolution of a module M is an injective resolution $(E^\cdot, \partial^\cdot)$ so that $E^i = E_A(Z^i(E^\cdot))$ for all i.

(7.65) Remark: Let M be an A-module. Then

(a) M has a minimal injective resolution $E^\cdot$

(b) $E^\cdot$ is unique up to isomorphism.

(c) If $I^\cdot$ is an injective resolution of M, then $E^\cdot$ is isomorphic to a direct summand of $I^\cdot$.

(7.66) <u>Definition</u>: Let A be a Noetherian ring and M a finitely generated A-module. For $P \in \mathrm{Spec}(A)$: $\mu_i(P,M) = \dim_{k(P)} \mathrm{Ext}^i_{A_P}(k(P), M_P)$ is called the i-th <u>Bass number</u> of M with respect to P.

(7.67) <u>Remark</u>: The Bass numbers $\mu_i(P,M)$ are finite, as can be seen by taking a free A-resolution $F_\cdot$ of $k(P)$ where all F_j are finite.

(7.68) <u>Theorem</u>: Let A be a Noetherian ring and M a finitely generated A-module. If $E^\cdot$ is a minimal injective A-resolution of M then $E^i \cong \bigoplus_P E(A/P)^{\mu_i(P,M)}$, where P runs over $\mathrm{Spec}(A)$.

<u>Proof</u>: By (7.63) we have to show that $\dim_{k(P)} \mathrm{Hom}_{A_P}(k(P), E^i_P) = \mu_i(P,M)$ for every prime ideal $P \in \mathrm{Spec}(A)$. Fix $P \in \mathrm{Spec}(A)$. Since $E^\cdot_P$ is a minimal injective A_P-resolution of M_P (Homework), we may replace A by A_P. Write m for the maximal ideal of A and k for A/m. It suffices to show $\mathrm{Hom}_A(k, E^i) \cong \mathrm{Ext}^i_A(k, M)$. Since $\mathrm{Ext}^i_A(k,M) = H^i(\mathrm{Hom}_A(k, E^\cdot))$, this will follow once we have shown that the differential on $\mathrm{Hom}_A(k, E^\cdot)$ is trivial. Note that $\mathrm{Hom}_A(k, E^\cdot) \cong C^\cdot$ where $C^\cdot$ is the subcomplex of $E^\cdot$ with $C^i = 0 :_{E^i} m$. If $0 \to M \xrightarrow{\partial^{-1}} E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \to \dots$ then $\partial^i(C^i) = 0$ for all $i \geq 0$ if $C^i \subseteq \mathrm{im}\,\partial^{i-1}$. Let $x \in C^i$. Since the extension $\mathrm{im}\,\partial^{i-1} \subseteq E^i$ is essential, there is an $a \in A$ with $0 \neq ax \in \mathrm{im}\,\partial^{i-1}$. As $mx = 0$ it follows that $a \in A - m = A^*$ and $x \in \mathrm{im}\,\partial^{i-1}$.