CHAPTER X: GORENSTEIN RINGS, MATLIS DUALITY

§1: GORENSTEIN RINGS

(10.1) Definition: Let (A, m, k) be a local Artinian ring. The socle $\mathcal{S}(A)$ of A is defined by $\mathcal{S}(A) = \operatorname{ann}(m) = \{a \in A \mid ma = 0\}$.

(10.2) Remark: If (A, m, k) is a local Artinian ring, then $\mathcal{S}(A)$ is an ideal of A. Since $\operatorname{Ass}_A(A) = \{m\}$, $\mathcal{S}(A) \neq (0)$ if $m \neq 0$. If $I \subseteq A$ is a nonzero ideal of A then $\operatorname{Ass}_A(I) = \{m\}$ and $I \cap \mathcal{S}(A) \neq (0)$. Moreover, $\mathcal{S}(A)$ is a finite-dimensional k-vector space.

(10.3) Proposition: Let (A, m, k) be a local Noetherian CM-ring of dimension d and let $x_1, \dots, x_d \in m$ and $y_1, \dots, y_d \in m$ be maximal regular sequences of A. Then $\dim_k(\mathcal{S}(A/(x_1, \dots, x_d))) = \dim_k(\mathcal{S}(A/(y_1, \dots, y_d)))$.

Proof: By induction on d: If $d = 0$, there is nothing to show. If $d = 1$, let $x, y \in A$ be regular elements of A. Then xy is regular and it suffices to show that $\dim_k(\mathcal{S}(A/(x))) = \dim_k(\mathcal{S}(A/(xy)))$. Let $a \in A - (x)$ with $ma \subseteq (x)$. Since y is regular, $ay \notin (xy)$ and $may \subseteq (xy)$. Thus multiplication by y defines an injective k-linear map: $\sigma: \mathcal{S}(A/(x)) \longrightarrow \mathcal{S}(A/(xy))$ with $\sigma(a + (x)) = ay + (xy)$. We claim that σ is surjective. Let $b \in A - (xy)$ with $mb \subseteq (xy)$. In particular, $xb \in (xy)$ and thus $b \in (y)$ since x is regular. Hence $b = yt$ with a unique $t \in A$. Since y is regular, $mty \subseteq (xy)$ implies $mt \subseteq (x)$ and $t + (x) \in \mathcal{S}(A/(x))$ with $\sigma(t + (x)) = yt + (xy) = b + (xy)$. σ is an isomorphism of k-vector spaces.

For the induction step suppose that the statement has been shown for local CM-rings of dimension $< d$. Let A be a local CM-ring of dimension d and let $x_1, \dots, x_d$ and $y_1, \dots, y_d$ be maximal regular sequences of A. Since x_d is regular on $A/(x_1, \dots, x_{d-1})$ and y_d regular on $A/(y_1, \dots, y_{d-1})$, $\operatorname{Ass}_A(A/(x_1, \dots, x_{d-1})) \cup \operatorname{Ass}_A(A/(y_1, \dots, y_{d-1})) =$

$\{P_1, \dots, P_s\}$ with $m \neq P_i$ for $1 \le i \le s$. Let $c \in m - (P_1 \cup \dots \cup P_s)$. Then c is a regular element of $A/(x_1, \dots, x_{d-1})$ and $A/(y_1, \dots, y_{d-1})$ and $x_1, \dots, x_{d-1}, c$ and $y_1, \dots, y_{d-1}, c$ are regular sequences. Hence $c, x_1, \dots, x_{d-1}$ and $c, y_1, \dots, y_{d-1}$ are regular sequences of A. Let $\overline{A} = A/(c)$. Then $\overline{A}$ is a CM-ring of dimension $d-1$ with regular sequences $x_1, \dots, x_{d-1}$ and $y_1, \dots, y_{d-1}$. By induction hypothesis:

$$\dim_k(\mathcal{I}(\overline{A}/(x_1, \dots, x_{d-1}))) = \dim_k(\mathcal{I}(\overline{A}/(y_1, \dots, y_{d-1}))).$$

Let $A' = A/(x_1)$. Then again by induction hypothesis:

$$\dim_k(\mathcal{I}(A'/(x_2, \dots, x_d))) = \dim_k(\mathcal{I}(A'/(x_2, \dots, x_{d-1}, c))).$$

This implies:

$$\begin{aligned} \dim_k(\mathcal{I}(A/(x_1, \dots, x_d))) &= \dim_k(\mathcal{I}(A/(x_1, \dots, x_{d-1}, c))) \\ &= \dim_k(\mathcal{I}(A/(y_1, \dots, y_{d-1}, c))) \\ &= \dim_k(\mathcal{I}(A/(y_1, \dots, y_d))). \end{aligned}$$

where the last equality follows by a similar argument as above.

(10.4) <u>Definition</u>: Let (A, m, k) be a local Noetherian CM-ring. The number $r = \dim_k(\mathcal{I}(A/(x_1, \dots, x_d)))$, where $x_1, \dots, x_d$ is a SOP of A, is called the <u>CM-type</u> of A. A is called a <u>Gorenstein ring</u> if $r = 1$. A Noetherian ring A is a <u>Gorenstein ring</u> if A_m is Gorenstein for all maximal ideals $m \subseteq A$.

(10.5) <u>Definition</u>: Let A be a Noetherian ring. An ideal $I \subseteq A$ is called <u>irreducible</u> if for all ideals $K, J \subseteq A$ with $I = K \cap J$ it follows that $I = K$ or $I = J$.

(10.6) <u>Lemma</u>: Let (A, m, k) be a local Noetherian ring and $Q \subseteq A$ an m-primary ideal. Q is irreducible if and only if $\dim_k(\mathcal{I}(A/Q)) = 1$.

<u>Proof</u>: Obviously, Q is irreducible in A if and only if (0) is irreducible in A/Q. Thus we may assume that A is a local Artinian ring and have to show that $(0) \subseteq A$ is irreducible if and only if $\dim_k(\mathcal{I}(A)) = 1$. If $(0) = K \cap J$ is reducible with $K \neq (0)$ and $J \neq (0)$, then $K \cap \mathcal{I}(A) \neq (0)$ and $J \cap \mathcal{I}(A) \neq (0)$ and $\dim_k(\mathcal{I}(A)) > 1$. Conversely,

if $\dim_k(\mathcal{I}(A)) \geq 2$, let $K, J \subseteq \mathcal{I}(A)$ be two nonzero subspaces with $K \cap J = (0)$. K and J are ideals of A and (0) is reducible.

(10.7) <u>Definition</u>: Let (A, m) be a local Noetherian ring. An ideal $I \subseteq A$ is called a <u>parameter ideal</u> if I is generated by a system of parameters of A.

(10.8) <u>Corollary</u>: Let (A, m) be a local CM-ring. The following are equivalent:

(a) A is Gorenstein

(b) A contains an irreducible parameter ideal.

(c) Every parameter ideal of A is irreducible.

Recall from chapter VII: If A is a ring and M an A-module, then

$$\operatorname{injdim} M \leq n \iff \operatorname{Ext}^i_A(N, M) = 0 \text{ for all } i > n \text{ and every } A\text{-module } N$$
$$\iff \operatorname{Ext}^{n+1}_A(A/I, M) = 0 \text{ for every } A\text{-ideal } I \quad (\text{see } (7.42)).$$

(10.9) <u>Lemma</u>: Let A be a Noetherian ring and M an A-module. Then $\operatorname{injdim} M \leq n$ if and only if $\operatorname{Ext}^{n+1}_A(A/P, M) = 0$ for every prime ideal $P \subseteq A$.

<u>Proof</u>: "$\Leftarrow$" It suffices to show that $\operatorname{Ext}^{n+1}_A(N, M) = 0$ for every finitely generated A-module N. Since A is Noetherian there is a decreasing chain of submodules of N: $N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_s \supseteq N_{s+1} = (0)$ so that $N_j/N_{j+1} \cong A/P_j$ for $0 \leq j \leq s$ where $P_j \subseteq A$ are prime ideals. We show by decreasing induction that $\operatorname{Ext}^{n+1}_A(N_j, M) = 0$. By assumption $\operatorname{Ext}^{n+1}_A(N_s, M) = 0$. Suppose that $\operatorname{Ext}^{n+1}_A(N_j, M) = 0$ for some $0 < j \leq s$. The exact sequence $0 \to N_j \to N_{j-1} \to N_{j-1}/N_j \to 0$ yields an exact sequence:

$$\begin{array}{ccccc} \operatorname{Ext}^{n+1}_A(N_{j-1}/N_j, M) & \longrightarrow & \operatorname{Ext}^{n+1}_A(N_{j-1}, M) & \longrightarrow & \operatorname{Ext}^{n+1}_A(N_j, M) \\ \cong & & & & \| \\ \operatorname{Ext}^{n+1}_A(A/P_{j-1}, M) = 0 & & & & 0 \text{ by ind. hyp.} \end{array}$$

Thus $\operatorname{Ext}^{n+1}_A(N_{j-1}, M) = 0$.

(10.10) <u>Lemma</u>: Let A be a Noetherian ring, M an A-module, and N a finitely generated A-module. If $n \in \mathbb{N}$ with $\mathrm{Ext}^n_A(A/P, M) = 0$ for all $P \in \mathrm{Supp}(N)$, then $\mathrm{Ext}^n_A(N, M) = 0$.

<u>Proof</u>: Homework (use a similar argument as in (10.9))

(10.11) <u>Proposition</u>: Let (A, m) be a Noetherian local ring, M a finitely generated A-module, and $P \in \mathrm{Spec}(A)$ with $P \neq m$. If $\mathrm{Ext}^{n+1}_A(A/Q, M) = 0$ for every prime ideal $Q \supsetneq P$, then $\mathrm{Ext}^n_A(A/P, M) = 0$.

<u>Proof</u>: Let $x \in m - P$. There is an exact sequence $0 \to A/P \xrightarrow{x} A/P \to N = A/(P, x) \to 0$. For every $Q \in \mathrm{Supp}(N)$ we have $P \subsetneq Q$ and thus $\mathrm{Ext}^{n+1}_A(A/Q, M) = 0$. By (10.10) $\mathrm{Ext}^{n+1}_A(N, M) = 0$. From the long exact sequence we obtain an exact sequence $\mathrm{Ext}^n_A(A/P, M) \xrightarrow{x} \mathrm{Ext}^n_A(A/P, M) \to \mathrm{Ext}^{n+1}_A(N, M) = 0$. Thus $\mathrm{Ext}^n_A(A/P, M) = 0$ by Nakayama's Lemma.

(10.12) <u>Proposition</u>: Let (A, m, k) be a local Noetherian ring and $M \neq 0$ a finitely generated A-module. Then $\mathrm{injdim}\, M = \sup\{n \mid \mathrm{Ext}^n_A(k, M) \neq 0\}$.

<u>Proof</u>: Use (10.9) and (10.11).

(10.13) <u>Corollary</u>: Let (A, m, k) be a local Noetherian ring, $M \neq 0$ a finitely generated A-module with $\mathrm{injdim}\, M < \infty$, and N a finitely generated A-module with $\mathrm{depth}\, N = 0$. Then $\mathrm{injdim}\, M = \sup\{n \mid \mathrm{Ext}^n_A(N, M) \neq 0\}$.

<u>Proof</u>: It suffices to show that if $t = \mathrm{injdim}\, M < \infty$, then $\mathrm{Ext}^t_A(N, M) \neq 0$. Since $\mathrm{depth}\, N = 0$, we have $k = A/m \hookrightarrow N$ giving an exact sequence $0 \to k \to N \to U \to 0$. Thus $\mathrm{Ext}^t_A(N, M) \to \mathrm{Ext}^t_A(k, M) \to \mathrm{Ext}^{t+1}_A(U, M)$ is exact. Since $t = \mathrm{injdim}\, M$, $\mathrm{Ext}^{t+1}_A(U, M) = 0$ and $\mathrm{Ext}^t_A(k, M) \neq 0$ by (10.12). Thus $\mathrm{Ext}^t_A(N, M) \neq 0$.

(10.14) <u>Lemma</u>: Let A be a ring, M and N A-modules, and $x \in \mathrm{ann}(N)$ a NZD on A and M. Then for all $n \geq 0$ $\mathrm{Ext}_A^{n+1}(N,M) \cong \mathrm{Ext}^n_{A/(x)}(N, M/xM)$. The isomorphism is natural in the first variable.

<u>Proof</u>: Let $F = \mathrm{Hom}_A(-, M/xM) : A/(x)\text{-mod} \longrightarrow A/(x)\text{-mod}$. We want to show that $R^nF \cong \mathrm{Ext}_A^{n+1}(-,M)$ as contravariant functors on $A/(x)$-mod.

(1) $F \cong \mathrm{Ext}^1_A(-,M)$. In order to prove this, consider the exact sequence $0 \to M \xrightarrow{x} M \longrightarrow M/(x)M \to 0$ and let N be an $A/(x)$-module. Since $\mathrm{Hom}_A(N,M)=0$ we have an exact sequence: $0 \longrightarrow \mathrm{Hom}_A(N, M/xM) \longrightarrow \mathrm{Ext}^1_A(N,M) \xrightarrow{x=0} \mathrm{Ext}^1_A(N,M)$.

(2) There is a longexact sequence $\mathrm{Ext}^\bullet_A(-,M)$ which is natural in the first variable.

(3) Let P be a free $A/(x)$-module. Since x is a NZD on A, $\mathrm{projdim}_A P \leq 1$ and $\mathrm{Ext}_A^{n+1}(P,M)=0$ for all $n+1 \geq 2$.

It follows now by induction on n that $R^nF \cong \mathrm{Ext}_A^{n+1}(-,M)$.

(10.15) <u>Proposition</u>: Let A be a local Noetherian ring, M a finitely generated A-module, and x a regular element on A and on M. Then $\mathrm{injdim}_{A/(x)} M/xM = \mathrm{injdim}_A M - 1$.

<u>Proof</u>: Use (10.12) and (10.14).

(10.16) <u>Remark</u>: We show in the next chapter: If A is a local Noetherian ring and $M \neq 0$ a finitely generated A-module of finite injective dimension, then $\dim M \leq \mathrm{injdim}\, M = \mathrm{depth}\, A$.

(10.17) <u>Lemma</u>: Let (A, m, k) be a local Noetherian ring, M a finitely generated A-module, and $P \subseteq m$ a prime ideal with $\mathrm{ht}(m/P) = d$. If $\mathrm{Ext}_A^{i+d}(k,M)=0$, then $\mathrm{Ext}^i_{A_P}(k(P), M_P)=0$ where $k(P) = (A/P)_P$.

<u>Proof</u>: By (10.11) $\mathrm{Ext}^i_A(A/P, M)=0$ and by (7.90)(b) $\mathrm{Ext}^i_{A_P}(k(P), M_P) \cong \mathrm{Ext}^i_A(A/P, M)_P = 0$.

(10.18) <u>Theorem</u>: Let (A, m, k) be a local Noetherian ring of dimension n. The following conditions are equivalent:

(a) $\operatorname{injdim} A < \infty$

(a') $\operatorname{injdim} A = n$

(b) $\operatorname{Ext}^i_A(k, A) = 0$ for $i \neq n$ and $\operatorname{Ext}^n_A(k, A) \cong k$

(c) $\operatorname{Ext}^i_A(k, A) = 0$ for some $i > n$

(d) $\operatorname{Ext}^i_A(k, A) = 0$ for $i < n$ and $\operatorname{Ext}^n_A(k, A) \cong k$

(d') A is a CM-ring and $\operatorname{Ext}^n_A(k, A) \cong k$

(e) A is a CM-ring and every parameter ideal is irreducible

(e') A is a CM-ring and there is an irreducible parameter ideal.

(f) A is a Gorenstein ring

<u>Proof</u>: By (10.8): (e) $\Leftrightarrow$ (e') $\Leftrightarrow$ (f). We will show: (a) $\Rightarrow$ (a') $\rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) and (b) $\rightarrow$ (d) $\leftrightarrow$ (d') $\Rightarrow$ (e) $\Rightarrow$ (b).

(a) $\Rightarrow$ (a'): Suppose that $\operatorname{injdim} A = r < \infty$. Let $P \subseteq A$ be a minimal prime ideal with $\operatorname{ht}(m/P) = n$. Then $PA_P \in \operatorname{Ass}(A_P)$ and $\operatorname{Hom}_{A_P}(k(P), A_P) \neq 0$. By (10.17) $\operatorname{Ext}^n_A(k, M) \neq 0$ and by (10.12) $r \geq n$. In order to show $r \leq n$ we proceed by induction on r. If $r = 0$, we are done. Since $\operatorname{injdim} A = r$, the functor $T = \operatorname{Ext}^r_A(-, A)$ is right exact by (7.42). Moreover, by (10.12) $\operatorname{Ext}^r_A(k, A) \neq 0$. If $m \in \operatorname{Ass}(A)$, the exact sequence $0 \to k \to A$ yields an exact sequence $\operatorname{Ext}^r_A(A, A) \longrightarrow \operatorname{Ext}^r_A(k, A) \longrightarrow 0$. Since $\operatorname{Ext}^r_A(k, A) \neq 0$, we obtain that $\operatorname{Ext}^r_A(A, A) \neq 0$, a contradiction since $r > 0$. Hence $m \notin \operatorname{Ass}(A)$ and there is a regular element $x \in m$. By (10.15) $B = A/(x)$ is a local Noetherian ring of injective dimension $r-1$. By induction hypothesis, $r - 1 \leq \dim B \leq n-1$ and $r \leq n$.

(a') $\Rightarrow$ (b): By induction on n. If $n = 0$, then $m \in \operatorname{Ass}(A)$ and there is an exact sequence $0 \to k \to A$. Since $\operatorname{injdim} A = 0$, the sequence $A \cong \operatorname{Hom}_A(A, A) \longrightarrow \operatorname{Hom}_A(k, A) \longrightarrow 0$ is $\operatorname{Hom}_A(k, A) \neq 0$ is cyclic and hence $\operatorname{Hom}_A(k, A) \cong k$.

If $n > 0$, the same argument as in (a) $\Rightarrow$ (a') yields that A contains a regular element $x \in m$. By (10.15) $B = A/(x)$ is a local Noetherian ring of injective dimension $n-1$. Thus by induction hypothesis and (10.14) $\mathrm{Ext}_A^{i+1}(k,A) \cong \mathrm{Ext}_B^i(k,B) = 0$ for all $i \neq n-1$ and $\mathrm{Ext}_R^n(k,A) \cong \mathrm{Ext}_B^{n-1}(k,B) \cong k$. Since x is a regular element on A, $\mathrm{Hom}_A(k,A) = 0$.

(b) $\Rightarrow$ (c): trivial

(c) $\Rightarrow$ (a): By induction on n: If $n=0$, let $\mathrm{Ext}_A^i(k,A) = 0$ for some $i>0$. Since m is the only prime ideal of A, by (10.9) $\mathrm{injdim}\, A \leq i < \infty$.

Let $n>0$ and $i>n$ with $\mathrm{Ext}_A^i(k,A) = 0$. We want to show that $\mathrm{Ext}_A^i(A/P,A) = 0$ for every prime ideal $P \subseteq A$. Then by (10.9) $\mathrm{injdim}\, A \leq i$. Assume that there is a prime ideal $P \subseteq A$ with $\mathrm{Ext}_A^i(A/P,A) \neq 0$. Since A is Noetherian, we may assume that $P \in \mathrm{Spec}(A)$ is maximal with $\mathrm{Ext}_A^i(A/P,A) \neq 0$. Then $P \neq m$ and for $x \in m - P$ consider the exact sequence $0 \to A/P \xrightarrow{x} A/P \to A/P+(x) \to 0$. This yields an exact sequence:

$$\mathrm{Ext}_A^i(A/P+(x),A) \longrightarrow \mathrm{Ext}_A^i(A/P,A) \xrightarrow{x} \mathrm{Ext}_A^i(A/P,A).$$

By assumption $\mathrm{Ext}_A^i(A/Q,A) = 0$ for all $Q \in \mathrm{Supp}(A/P+(x))$. Thus by (10.10) $\mathrm{Ext}_A^i(A/P+(x),A) = 0$ and x is regular on $\mathrm{Ext}_A^i(A/P,A)$. On the other hand, if $\mathrm{ht}(m/P) = d$, then by (10.17) $\mathrm{Ext}_{A_P}^{i-d}(k(P),A_P) = 0$. Since $\dim A_P \leq n-d < i-d$, by induction hypothesis $\mathrm{injdim}\, A_P < \infty$ and by '(a) $\Rightarrow$ (b)': $\mathrm{Ext}_{A_P}^i(k(P),A_P) \cong \mathrm{Ext}_A^i(A/P,A)_P = 0$. Since $\mathrm{Ext}_A^i(A/P,A)$ is a finitely generated A-module, there is an element $x \in m - P$ with $x\,\mathrm{Ext}_A^i(A/P,A) = 0$. Thus $\mathrm{Ext}_A^i(A/P,A) = 0$.

(b) $\Rightarrow$ (d): trivial

(d) $\Leftrightarrow$ (d'): Theorem (8.16)

(d') $\Rightarrow$ (f): Let $x_1, \ldots, x_n$ be an SoP of A, $I = (x_1, \ldots, x_n)$, and $B = A/I$. Since A is a CM-ring, $x_1, \ldots, x_n$ is a regular sequence of A. Repeated application of (10.14) yields:

$\mathrm{Ext}_A^n(k,A) \cong \mathrm{Ext}_{A/(x_1)}^{n-1}(k, A/(x_1)) \cong \ldots \cong \mathrm{Ext}_B^0(k,B) = \mathrm{Hom}_B(k,B) \cong k$.

Thus $\dim_k(\mathcal{Y}(B)) = 1$ and A is a Gorenstein ring.

(f) $\Rightarrow$ (b): Since A is a CM-ring, by (8.16) $\mathrm{Ext}_A^i(k,A) = 0$ for $i<n$. If $I = (x_1, \ldots, x_n) \subseteq A$ is a parameter ideal of A and $B = A/I$, then by (10.14) $\mathrm{Ext}_A^n(k,A) \cong \mathrm{Ext}_B^0(k,B) = \mathrm{Hom}_B(k,B)$ and $\mathrm{Hom}_B(k,B) \cong k$, since A is Gorenstein.

In order to show that $\mathrm{Ext}_A^i(k,A) = 0$ for all $i>n$, let I and B be as above.

By (10.14) it suffices to show that $\mathrm{Ext}^i_B(k,B)=0$ for $i>0$. Since $\dim B=0$, by (10.9) $\mathrm{Ext}^1_B(k,B)=0$ implies that $\mathrm{injdim}\, B\leq 1$ and it suffices to show that $\mathrm{Ext}^1_B(k,B)=0$. Let $B=N_r\supseteq N_{r-1}\supseteq\cdots\supseteq N_1\supseteq N_0=0$ be a descending chain of ideals with $N_i/N_{i-1}\cong k$ for all $1\leq i\leq r$. This yields short exact sequences: $0\to N_i\to N_{i+1}\to k\to 0$ for all $1\leq i\leq r-1$ inducing longexact sequences:

$$0\to \mathrm{Hom}_B(k,B)\to \mathrm{Hom}_B(N_{i+1},B)\to \mathrm{Hom}_B(N_i,B)\xrightarrow{\delta_i}\mathrm{Ext}^1_B(k,B).$$

Since B is Gorenstein, $\mathrm{Hom}_B(k,B)\cong\mathrm{Hom}_B(N_1,B)\cong k$ and by induction on i $\ell_B(\mathrm{Hom}_B(N_i,B))\leq i$ for all $1\leq i\leq r$. Moreover, $\ell_B(\mathrm{Hom}_B(N_i,B))=i \iff \delta_j=0$ for all $j\leq i$. Since $\ell_B(\mathrm{Hom}_B(B,B))=\ell_B(B)=r$ it follow that $\delta_1=\delta_2=\cdots=\delta_{r-1}=0$. Thus the exact sequence $0\to N_{r-1}\to N_r=B\to 0$ yields a longexact sequence $0\to\mathrm{Ext}^1_B(k,B)\to\mathrm{Ext}^1_B(B,B)\to\ldots\ldots$ Since $\mathrm{Ext}^1_B(B,B)=0$, $\mathrm{Ext}^1_B(k,B)=0$ and $\mathrm{injdim}\, B\leq 1$.

(10.19) <u>Corollary</u>: Let (A,m,k) be a local Noetherian CM-ring of CM-type r and dimension n Then $r=\dim_k(\mathrm{Ext}^n_A(k,A))$.

<u>Proof</u>: Let $I=(x_1,\text{—},x_n)$ be a parameter ideal of A and $B=A/I$. By (10.14): $\mathrm{Ext}^n_A(k,A)\cong\mathrm{Hom}_B(k,B)$ and $r=\dim_k(\mathcal{J}(B))=\dim_k(\mathrm{Hom}_B(k,B))$.

(10.20) <u>Theorem:</u> Let (A,m,k) be a local Gorenstein ring and $P\in\mathrm{Spec}(A)$. Then A_P is a local Gorenstein ring.

<u>Proof</u>: Consider a finite injective resolution of A: $0\to A\to E_0\to E_1\to\cdots\to E_n\to 0$. By Homework 5, #5: $(E_i)_P$ is an injective A_P-module and $0\to A_P\to E_{0P}\to\cdots\to E_{nP}\to 0$ is a finite injective resolution of A_P. Thus $\mathrm{injdim}\, A_P<\infty$.

(10.21) <u>Theorem</u>: Let (A,m,k) be a local Noetherian ring and $\hat{A}$ the m-adic completion of A. A is Gorenstein if and only if $\hat{A}$ is Gorenstein.

Proof: By (9.53) A is CM if and only if $\hat{A}$ is CM. Let $I=(x_1,\ldots,x_n)$ be a parameter ideal of A. Then $I\hat{A}$ is a parameter ideal of $\hat{A}$ and $A/I \cong \hat{A}/I\hat{A}$.

Recall from Chapter 7:

Let A be a Noetherian ring, M a finitely generated A-module, and $E^\bullet$ a minimal injective resolution of M. Then $E^i \cong \bigoplus_{P\in \operatorname{Spec}(A)} E(A/P)^{\mu_i(P,M)}$, where $\mu_i(P,M) = \dim_{k(P)} \operatorname{Ext}^i_{A_P}(k(P), M_P)$ is the ith Bass number of M with respect to P.

(10.21A) Theorem: Let A be a Noetherian ring and $E^\bullet$ a minimal injective resolution of A. A is Gorenstein if and only if for all $i\geq 0$: $E^i \cong \bigoplus_{\operatorname{ht} P=i} E(A/P)$ or equivalently, for all $P\in \operatorname{Spec}(A)$: $\mu_i(P,A)=\delta_{i,\operatorname{ht} P}$.

Proof: "$\Leftarrow$": Suppose that $\mu_i(P,A)=\delta_{i,\operatorname{ht} P}$ for all $i\geq 0$ and all $P\in\operatorname{Spec}(A)$. By Homework 5, #5 for all $P\in \operatorname{Spec}(A)$, $E^\bullet_P$ is a minimal injective resolution of A_P. Since $E^\bullet_P$ is finite, $\operatorname{injdim}(A_P)<\infty$ and A_P is Gorenstein.
"$\Rightarrow$": Let $P\in\operatorname{Spec}(A)$. By assumption A_P is Gorenstein, thus by (10.18) $\operatorname{Ext}^i_{A_P}(k(P),A_P)=0$ for $i\neq \dim A_P = \operatorname{ht} P$ and $\operatorname{Ext}^i_{A_P}(k(P),A_P)\cong k(P)$ if $i=\dim A_P=\operatorname{ht} P$. Thus $\mu_i(P,A)=\delta_{i,\operatorname{ht} P}$.

§2: MATLIS DUALITY

Recall from Chapter VII:

Let A be a Noetherian ring and E an injective A-module. Then

$$E \cong \bigoplus_{P \in \mathrm{Spec}(A)} E(A/P)^{\mu_P}$$

where $E(A/P)$ is the injective hull of A/P and $\mu_P = \dim_{k(P)} \mathrm{Hom}_{A_P}(k(P), E_P)$ (7.65).

Let (A, m, k) be a local Noetherian ring and $E = E(k)$ the injective hull of k. For an A-module M set $M' = \mathrm{Hom}_A(M, E)$. Then $M'' = \mathrm{Hom}_A(M,E)' = \mathrm{Hom}_A(\mathrm{Hom}_A(M,E), E)$ and there is a canonical map $\theta : M \longrightarrow M'' = \mathrm{Hom}_A(\mathrm{Hom}_A(M,E), E)$ defined by: for $x \in M$, $\theta(x) : \mathrm{Hom}_A(M,E) \longrightarrow E$ is given by $\theta(x)(\varphi) = \varphi(x)$ for all $\varphi \in \mathrm{Hom}_A(M,E)$. Note that θ is A-linear.

(10.22) Proposition: Assumptions as above and suppose $M \neq 0$.

(a) For all $x \in M - (0)$ there is an $\varphi \in M'$ with $\varphi(x) \neq 0$. In particular, φ is injective.

(b) If M is an A-module of finite length then $\ell_A(M) = \ell_A(M')$ and θ is an isomorphism.

Proof: (a) The submodule Ax of M is isomorphic to $A/\mathrm{ann}(x)$. Let f be the composition of maps: $f : Ax \xrightarrow{\cong} A/\mathrm{ann}(x) \xrightarrow{\mathrm{can}} k \hookrightarrow E$. Then $f(x) \neq 0$. Since E is injective, f extends to an A-linear map $\varphi : M \longrightarrow E$ with $\varphi(x) \neq 0$.

(b) Let $M_1 \subseteq M$ be a submodule with $\ell_A(M_1) = n - 1 = \ell_A(M) - 1$. The exact sequence $0 \longrightarrow M_1 \longrightarrow M \longrightarrow k \longrightarrow 0$ yields an exact sequence $0 \longrightarrow k' \longrightarrow M' \longrightarrow M_1' \longrightarrow 0$. Since E is an essential extension of k: $k' = \mathrm{Hom}_A(k, E) \cong \mathrm{Hom}_A(k,k) \cong k$ and thus $\ell_A(M') = \ell_A(M_1') + 1$. The statement follows by induction on $n = \ell_A(M)$.

(10.23) Proposition: Assumptions as above. Let $\hat{A}$ be the m-adic completion of A.

(a) E is an $\hat{A}$-module. Moreover, E is the injective hull of the $\hat{A}$-module k.

(b) $\mathrm{Hom}_A(E,E) = \mathrm{Hom}_{\hat{A}}(E,E) \cong \hat{A}$.

<u>Proof</u>: (a) We claim that the canonical map $\sigma: E \longrightarrow E\otimes_A \hat{A}$ defined by $\sigma(x) = x\otimes 1$ is an isomorphism. Let $x\otimes\hat{a} \in E\otimes_A\hat{A}$. By (7.59) there is an $n\in\mathbb{N}$ with $m^n x = 0$. Since A is dense in $\hat{A}$, there is an $a_0\in A$ so that $\hat{a} - a_0 \in \widehat{m^n A}$. Thus $\hat{a} = a_0 + \sum_{i=1}^{t} y_i\hat{b}_i$ where $y_i \in m^n \subseteq A$ and $\hat{b}_i \in \hat{A}$. Then $x\otimes\hat{a} = x\otimes(a_0 + \sum y_i\hat{b}_i) = x\otimes a_0 + x\otimes\sum y_i\hat{b}_i = a_0 x\otimes 1 + \sum y_i x\otimes b = a_0 x\otimes 1$. Thus σ is surjective. In order to show that σ is injective consider the following commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E\otimes_A\hat{A} \\ \varepsilon\uparrow & & \uparrow \delta = \varepsilon\otimes\hat{A} \\ k & \xrightarrow{\rho} & k\otimes_A\hat{A} \end{array}$$

ε and ρ are injective. Since $\hat{A}$ is flat over A, δ is injective. Thus $\sigma\varepsilon = \delta\rho$ is injective. Since E is an essential extension of k, σ is injective.

Let F be the injective hull of E as an $\hat{A}$-module. Then F is the injective hull of the $\hat{A}$-module k and by (7.59) every element of F is annihilated by some power of $m\hat{A}$. F is an A-module with $E\subseteq F$ and since E is injective over A, there is a submodule C of F with $F = E\oplus C$. Since every element of C is annihilated by some power of $m\hat{A}$, C is an $\hat{A}$-module. But F is indecomposable as $\hat{A}$-module. Thus $C=0$ and $F=E$.

(b) For $\nu>0$ set $E_\nu = \{x\in E \mid m^\nu x = 0\}$. E_ν is a module over A and $\hat{A}$, $E_\nu \subseteq E_{\nu+1}$, and $E = \bigcup E_\nu = \varinjlim E_\nu$. Thus by (7.91) $\mathrm{Hom}_A(E,E) = \mathrm{Hom}_A(\varinjlim E_\nu, E) = \varprojlim \mathrm{Hom}_A(E_\nu, E)$. Since $\mathrm{Hom}_A(A/m^\nu, E) \cong E_\nu$, $\mathrm{Hom}_A(E_\nu, E) = E_\nu' = (A/m^\nu)''$. A/m^ν is an A-module of finite length, thus by (10.22): $A/m^\nu \cong (A/m^\nu)''$. Thus $\varprojlim \mathrm{Hom}_A(E_\nu,E) \cong \varprojlim A/m^\nu \cong \hat{A}$ as A-modules. Since E_ν is also an $\hat{A}$-module, the same argument shows that $\mathrm{Hom}_{\hat{A}}(E,E) \cong \hat{A}$ as $\hat{A}$-modules.

(10.24) <u>Remark</u>: (10.23)(b) shows that every $\hat{A}$-linear map: $f: E\longrightarrow E$ is the multiplication by some $\hat{a}\in\hat{A}$.

(10.25) <u>Theorem</u>: Let (A, m, k) be a local Noetherian ring and $E = E_A(k)$ the injective hull of k. E is an Artinian module over A and $\hat{A}$.

<u>Proof</u>: Note that every A-submodule of E is also an $\hat{A}$-submodule. Thus we may assume that A is complete. If $M \subseteq E$ is a submodule let $M^\perp = \mathrm{ann}(M) = \{a \in A \mid aM = 0\}$ and if $I \subseteq A$ is an ideal set $I^\perp = 0 :_E I = \{x \in E \mid Ix = 0\}$. Obviously, $M \subseteq M^{\perp\perp}$. We want to show that $M^{\perp\perp} = M$. Consider the exact sequence $0 \to M \to E \to E/M \to 0$. Since E is injective, the sequence $0 \to (E/M)' \xrightarrow{(*)} E'$ is exact, where $(E/M)' = \mathrm{Hom}_A(E/M, E)$ and $E' = \mathrm{Hom}_A(E,E) \cong \hat{A}$ (10.23). Thus for every $f \in \mathrm{Hom}_A(E,E)$ there is an $\hat{a} \in \hat{A}$ so that $f(x) = \hat{a}x$ for all $x \in E$. Every $g \in \mathrm{Hom}_A(E/M, E)$ is mapped under $(*)$ into an $f \in \mathrm{Hom}_A(E,E)$ with $f|_M = 0$ and conversely every $f \in \mathrm{Hom}_A(E,E)$ with $f|_M = 0$ factors through a $g \in \mathrm{Hom}_A(E/M, E)$. Thus $\mathrm{Hom}_A(E/M, E) \cong M^\perp$ and the embedding $(*)$ corresponds to the embedding $0 \to M^\perp \hookrightarrow A$. By (10.22) for all $x \in E - M$ there is an $\varphi \in (E/M)'$ with $\varphi(x+M) \neq 0$. Thus for all $x \in E - M$ there is an $a \in M^\perp$ with $ax \neq 0$. This shows that $M^{\perp\perp} \subseteq M$ and hence $M^{\perp\perp} = M$.

If $I \subseteq A$ is an ideal, the exact sequence $0 \to I \to A \to A/I \to 0$ yields an exact sequence $0 \to (A/I)' \xrightarrow{(\tilde{*})} A' = \mathrm{Hom}_A(A, E) \cong E$. Under $(\tilde{*})$ $(A/I)'$ is mapped onto $I^\perp$. If $a \in A - I$, by (10.22) there is a $\varphi \in (A/I)'$ with $\varphi(a+I) \neq 0$. Let $x = \varphi(1+I) \in I^\perp$. Then $Ix = 0$ and $a \notin \mathrm{ann}(x)$. Since $I^{\perp\perp} = \bigcap_{x \in I^\perp} \mathrm{ann}(x)$ it follows that $I^{\perp\perp} \subseteq I$ and hence $I^{\perp\perp} \subseteq I$.

This shows that '$\perp$' defines order-reversing bijections between the sets:

$$\{M \mid M \subseteq E \text{ a submodule}\} \xrightarrow{\perp} \{I \mid I \subseteq A \text{ an ideal}\}.$$

Since A is Noetherian, E is Artinian.

(10.26) <u>Theorem</u>: Let (A, m, k) be a complete local Noetherian ring and $E = E_A(k)$ the injective hull of k.

(a) If M is a Noetherian A-module, then $M' = \mathrm{Hom}_A(M, E)$ is an Artinian A-module and $M'' \cong M$.

(b) If M is an Artinian A-module, then $M' = \mathrm{Hom}_A(M,E)$ is a Noetherian A-module and $M'' \cong M$.

Proof: (a) Let $n \in \mathbb{N}$ with $A^n \longrightarrow M \longrightarrow 0$ exact. Since E is injective, the sequence $0 \longrightarrow M' \longrightarrow (A^n)' = \mathrm{Hom}_A(A^n, E) \cong \mathrm{Hom}_A(A,E)^n \cong E^n$ is exact. E^n is an Artinian A-module and so is every submodule of E^n. Thus M' is Artinian. Since $(E^n)' = \mathrm{Hom}_A(E^n, E) \cong \mathrm{Hom}_A(E,E)^n \cong A^n$ there is a commutative diagram with exact rows:

$$\begin{array}{ccccc} A^n & \longrightarrow & M & \longrightarrow & 0 \\ \cong\downarrow\theta' & & \downarrow\theta & & \\ (E^n)'' & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

By (10.22) θ is injective, hence θ is an isomorphism.

(b) We claim that there is an $n \in \mathbb{N}$ so that M can be considered a submodule of E^n. For all $m \in \mathbb{N}$ consider all A-linear maps $\tau : M \longrightarrow E^m$. This yields a set Γ of submodules $\ker(\tau)$ of M. Since M is Artinian, there is an $n \in \mathbb{N}$ and an A-linear map $\varphi : M \longrightarrow E^n$ so that $\ker(\varphi)$ is minimal in Γ. If $\ker(\varphi) \neq 0$ and $x \in \ker(\varphi) - (0)$, by (10.22) there is an A-linear map $\sigma : M \longrightarrow E$ with $\sigma(x) \neq 0$. Let $\rho : M \longrightarrow E^{n+1}$ be defined by $\rho(y) = (\varphi(y), \sigma(y))$. Then $\ker(\rho) = \ker(\varphi) \cap \ker(\sigma) \subsetneq \ker(\varphi)$, a contradiction. Thus $\ker(\varphi) = 0$.

Let $0 \longrightarrow M \longrightarrow E^n$ be exact. Then $(E^n)' \longrightarrow M' \longrightarrow 0$ is exact and $(E^n)' \cong A^n$. M' is a Noetherian A-module.

In order to show that $M'' \cong M$ note that every homomorphic image of an Artinian module is Artinian. The exact sequence $0 \longrightarrow M \longrightarrow E^n \longrightarrow E^n/M \longrightarrow 0$ yields a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E^n & \longrightarrow & E^n/M & \longrightarrow & 0 \\ & & \theta\downarrow & & \downarrow\cong & & \downarrow\bar{\theta} & & \\ 0 & \longrightarrow & M'' & \longrightarrow & (E^n)'' & \longrightarrow & (E^n/M)'' & \longrightarrow & 0 \end{array}$$

By (10.22) θ and $\bar{\theta}$ are injective. Thus $\bar{\theta}$ is an isomorphism. Hence θ is an isomorphism by the 5-Lemma.

§3: THE CANONICAL MODULE

(10.27) Definition: Let (A, m, k) be a local Noetherian ring and M a finitely generated A-module with depth $M = d$. The number $r(M) = \dim_k \operatorname{Ext}_A^d(k, M)$ is called the type of M.

(10.28) Lemma: Let A be a Noetherian ring, M a finitely generated A-module and $x_1, \text{—}, x_n$ an M-sequence. Suppose that N is an A-module with $I = (x_1, \text{—}, x_n) \subseteq \operatorname{ann}(N)$. Then $\operatorname{Hom}_A(N, M/IM) \cong \operatorname{Ext}_A^n(N, M)$.

Proof: Set $M_0 = M$ and $M_i = M/(x_1, \text{—}, x_i)M$. We show by induction on i that $\operatorname{Ext}_A^{n-i-1}(N, M_{i+1}) \cong \operatorname{Ext}_A^{n-i}(N, M_i)$. If $i = 0$ the exact sequence $0 \to M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$ yields a longexact sequence:

$$\operatorname{Ext}_A^{n-1}(N, M) \longrightarrow \operatorname{Ext}_A^{n-1}(N, M_1) \longrightarrow \operatorname{Ext}_A^n(N, M) \xrightarrow{\varphi} \operatorname{Ext}_A^n(N, M)$$

where φ is multiplication by x_1. Since $x_1 \in \operatorname{ann}(N)$, $\varphi = 0$ and $\operatorname{Ext}_A^{n-1}(N, M) = 0$ by (8.14). For the induction step $i \Rightarrow i+1$ consider the exact sequence: $0 \to M_i \xrightarrow{x_{i+1}} M_i \longrightarrow M_{i+1} \longrightarrow 0$ and repeat the argument. Thus $\operatorname{Hom}_A(N, M_n) \cong \operatorname{Ext}_A^n(N, M)$.

(10.29) Remark: Let (A, m, k) be a local Noetherian ring, M a finitely generated A-module and $x_1, \text{—}, x_d$ a maximal M-sequence. Then $r(M) = \dim_k \operatorname{Hom}_A(k, M/IM)$ where $I = (x_1, \text{—}, x_d)$. $\operatorname{Hom}_A(k, M/IM) \cong 0 :_{M/IM} m$ is called the socle of M/IM.

Recall from Chapter VIII: Let A be a local Noetherian ring and M a finitely generated A-module. M is called a maximal CM-module (MCM) if depth $M = \dim A$

(10.30) Definition: Let A be a local CM-ring. A finitely generated A-module C is called a canonical module of A if C is a MCM, $r(C) = 1$, and $\operatorname{injdim}_A C < \infty$.

(10.31) <u>Examples</u>: Let (A, m, k) be a local Noetherian ring.

(a) If A is Gorenstein then A is a canonical module of A.

(b) If A is Artinian then $E_A(k)$ is a canonical module of A. Conversely, every canonical module of A is isomorphic to $E_A(k)$.

<u>Proof</u>: (b) The Artinian local ring A is complete. Thus by Matlis duality (10.26) $A' = \mathrm{Hom}_A(A, E_A(k)) \cong E_A(k)$ is a Noetherian A-module. Thus $E_A(k)$ is finitely generated. Conversely, if C is a canonical module of A, then $\mathrm{injdim}\, C \leq \mathrm{depth}\, A = 0$ (chapter $\underline{\mathrm{XI}}$) and C is injective. Now use (7.63).

(10.32) <u>Lemma</u>: Let A be a local Noetherian ring, $\varphi: M \longrightarrow N$ an A-linear map of finitely generated A-modules and $x_1, \dots, x_n$ an N-regular sequence. Let $I = (x_1, \dots, x_n)$. If $\varphi \otimes_A A/I$ is an isomorphism, then so is φ.

<u>Proof</u>: By Nakayama's Lemma φ is surjective. Thus there is an exact sequence $C_\bullet : 0 \longrightarrow U \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0$. Since $x_1, \dots, x_n$ is an N-sequence by Homework #6, problem 4, $C_\bullet \otimes_A A/I$ is exact. Thus $U \otimes_A A/I \cong \ker(\varphi \otimes_A A/I) = 0$ and $U = 0$ by Nakayama's Lemma.

(10.33) <u>Lemma</u>: Let A be a local Noetherian ring and M a MCM A-module. Every A-regular sequence is M-regular.

<u>Proof</u>: Obviously, $M \neq 0$. If $P \in \mathrm{Ass}_A(M)$ then by (8.21) $\dim A/P \geq \mathrm{depth}\, M = \dim A$ and P is a minimal prime of A. Hence $P \in \mathrm{Ass}(A)$ and every A-regular element x is M-regular. Since M/xM is a MCM $A/(x)$-module the assertion follows by induction.

(10.34) <u>Proposition</u>: Let A be a local CM ring of dimension d, M a finitely generated

A-module, and C a MCM A-module.

(a) If M is CM of dimension t and $\operatorname{injdim}_A C<\infty$ then $\operatorname{Ext}^i_A(M,C)=0$ for $i\neq d-t$ and $\operatorname{Ext}^{d-t}_A(M,C)$ is CM of dimension t.

(b) If $\operatorname{Ext}^i_A(M,C)=0$ for all $i>0$ then $\operatorname{depth}\operatorname{Hom}_A(M,C)\geq d$

(c) If $\operatorname{Ext}^i_A(M,C)=0$ for all $i>0$ and M is MCM and $x_1,\ldots,x_d$ is an A-regular sequence then $\operatorname{Hom}_A(M,C)\otimes_A A/(\underline{x})\cong \operatorname{Hom}_{A/(\underline{x})}(M/(\underline{x})M, C/(\underline{x})C)$ via the natural map.

Proof: (a) By (8.20) $\operatorname{Ext}^i_A(M,C)=0$ for all $i<\operatorname{depth} C-\dim M=d-t$. Since $\operatorname{ann}(M)\subseteq \operatorname{ann}(\operatorname{Ext}^i_A(M,C))$, $\dim \operatorname{Ext}^{d-t}_A(M,C)\leq \dim M=t$. We want to show by induction on t that $\operatorname{Ext}^i_A(M,C)=0$ for $i>d-t$ and that $\operatorname{depth}\operatorname{Ext}^{d-t}_A(M,C)=t$.
If $t=\operatorname{depth} M=0$, then by (10.13) $\operatorname{injdim} C=d=\max\{i \mid \operatorname{Ext}^i_A(M,C)\neq 0\}$. Hence $\operatorname{Ext}^i_A(M,C)=0$ for $i>d$ and $\operatorname{Ext}^i_A(M,C)\neq 0$. Moreover, $\operatorname{depth}\operatorname{Ext}^d_A(M,C)=0$ since $\dim \operatorname{Ext}^d_A(M,C)=0$. Let $t\geq 1$ and let $x\in m_A$ be an M-regular element. The exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$ yields a long exact sequence:

$$\cdots\to \operatorname{Ext}^i_A(M/xM,C)\to \operatorname{Ext}^i_A(M,C)\xrightarrow{x}\operatorname{Ext}^i_A(M,C)\to \operatorname{Ext}^{i+1}_A(M/xM,C)\to\cdots$$

M/xM is a CM-module of dimension $t-1$. For $i\neq d-t$, by induction hypothesis, $\operatorname{Ext}^{i+1}_A(M/xM,C)=0$, hence by Nakayama's Lemma $\operatorname{Ext}^i_A(M,C)=0$. For $i=d-t$ we have an exact sequence $0\to \operatorname{Ext}^{d-t}_A(M,C)\xrightarrow{x}\operatorname{Ext}^{d-t}_A(M,C)\to \operatorname{Ext}^{d-t+1}_A(M/xM,C)\to 0$. Thus $\operatorname{Ext}^{d-t+1}_A(M/xM,C)\cong \operatorname{Ext}^{d-t}_A(M,C)/x\operatorname{Ext}^{d-t}_A(M,C)$.
By induction hypothesis $\operatorname{depth}\operatorname{Ext}^{d-t+1}_A(M/xM,C)=t-1$ and hence $\operatorname{depth}\operatorname{Ext}^{d-t}_A(M,C)=t$.

(b) Let $F_\bullet$ be a finite free A-resolution of M. Since $\operatorname{Ext}^i_A(M,C)=0$ for all $i>0$, the sequence $0\to\operatorname{Hom}_A(M,C)\to\operatorname{Hom}_A(F_\bullet,C)$ is exact. This yields an exact sequence:

$(*)$ $0\to \operatorname{Hom}_A(M,C)\to\operatorname{Hom}_A(F_0,C)\to\cdots\to\operatorname{Hom}_A(F_{d-1},C)\to B_d\to 0$

with $\operatorname{Hom}_A(F_i,C)\cong C^{b_i}$, $\operatorname{depth}\operatorname{Hom}_A(F_i,C)=d$ and $\operatorname{depth} B_d\geq 1$. Splitting $(*)$ into short exact sequences $0\to B_i\to\operatorname{Hom}_A(F_i,C)\to B_{i+1}\to 0$ and applying (8.28) yields that $\operatorname{depth} B_i\geq \min\{d,\operatorname{depth} B_{i+1}+1\}$. Thus $\operatorname{depth}\operatorname{Hom}_A(M,C)=d$.

(c) Let $x \in m_A$ be A-regular. By (10.33) x is M- and C-regular and M/xM, C/xC are MCM over $A/(x)$. From the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ we obtain $0 = \operatorname{Ext}^i_A(M,C) \to \operatorname{Ext}^{i+1}_A(M/xM, C) \to \operatorname{Ext}^{i+1}_A(M,C) = 0$ and thus $\operatorname{Ext}^{i+1}_A(M/xM, C) = 0$ for all $i \geq 1$. By (10.14) $\operatorname{Ext}^i_{A/(x)}(M/xM, C/xC) \cong \operatorname{Ext}^{i+1}_A(M/xM, C) = 0$ for all $i \geq 1$ and the $A/(x)$-modules M/xM and C/xC satisfy the assumptions.

We proceed by induction on d. If $x_1 = x \in m_A$ is A-regular, the exact sequence $0 \to C \xrightarrow{x} C \to C/xC \to 0$ induces an exact sequence: $0 \to \operatorname{Hom}_A(M,C) \xrightarrow{x} \operatorname{Hom}_A(M,C) \to \operatorname{Hom}_A(M, C/xC) = \operatorname{Hom}_{A/(x)}(M/xM, C/xC) \to \operatorname{Ext}^1_A(M,C) = 0$.

Thus $\operatorname{Hom}_{A/(x)}(M/xM, C/xC) \cong \operatorname{Hom}_A(M,C) \otimes_A A/(x)$. This shows the case $d=1$.

If $d>1$, let $x_1, \ldots, x_d$ be an A-regular sequence. Then by induction hypothesis:

$$\begin{aligned}\operatorname{Hom}_{A/(\underline{x})}(M/(\underline{x}), C/(\underline{x})) &\cong \operatorname{Hom}_{A/(x_1)}(M/x_1M, C/x_1C) \otimes_{A/(x_1)} A/(\underline{x}) \\ &\cong \operatorname{Hom}_A(M,C) \otimes_A A/(\underline{x}).\end{aligned}$$

(10.35) <u>Theorem</u>: Let (A, m) be a local CM-ring of dimension d, $\underline{x} = x_1, \ldots, x_d$ an A-regular sequence, and C, C' canonical modules of A.

(a) $\underline{x}$ is regular on C and $C/(\underline{x})C \cong E_{A/(\underline{x})}(k)$

(b) $C \cong C'$; in particular, we can talk about 'the' canonical module of A, denoted by ω_A, provided it exists.

<u>Proof</u>: (a) By (10.33) $\underline{x}$ is regular on C. Thus $r(C/(\underline{x})) = r(C) = 1$ by (10.29) and $\operatorname{injdim}_{A/(\underline{x})} C/(\underline{x})C < \infty$ by (10.15). Hence $C/(\underline{x})C$ is a canonical module of the Artinian local ring $A/(\underline{x})$ and by (10.31) $C/(\underline{x})C \cong E_{A/(\underline{x})}(k)$.

(b) Let $\bar{A} = A/(\underline{x})$. C and C' are MCM modules and $\operatorname{injdim} C' < \infty$. By (10.34)(a) $\operatorname{Ext}^i_A(C, C') = 0$ for all $i > 0$ and by (10.34)(c) $\operatorname{Hom}_A(C, C') \otimes_A \bar{A} \cong \operatorname{Hom}_{\bar{A}}(C/(\underline{x})C, C'/(\underline{x})C')$ via the natural map. By (a) there is an isomorphism $\varphi \in \operatorname{Hom}_{\bar{A}}(C/(\underline{x})C, C'/(\underline{x})C')$. Thus there is a $\psi \in \operatorname{Hom}_A(C, C')$ with $\psi \otimes_A \bar{A} = \varphi$. By (10.32) ψ is an isomorphism since $\underline{x}$ is regular on C'.

(10.36) Theorem: Let A be a local Noetherian ring. The following are equivalent:
(a) A is Gorenstein
(b) A is CM, ω_A exists and $\omega_A \cong A$.

Proof: By (10.18) A is Gorenstein if and only if A is CM, $r(A)=1$, and $\mathrm{injdim}_A A < \infty$. The assertion follows.

(10.37) Theorem: Let A be a local CM ring of dimension d, and C a finitely generated A-module. The following are equivalent:

(a) $C \cong \omega_A$

(b) For every t, $0 \le t \le d$, and every CM A-module M of dimension t:
  (i) $\mathrm{Ext}_A^{d-t}(M,C)$ is a CM-module of dimension t
  (ii) $\mathrm{Ext}_A^i(M,C)=0$ for all $i \neq d-t$
  (iii) there is a natural isomorphism $M \xrightarrow{\sim} \mathrm{Ext}_A^{d-t}(\mathrm{Ext}_A^{d-t}(M,C),C)$

(c) For every MCM A-module M:
  (i) $\mathrm{Hom}_A(M,C)$ is a MCM A-module
  (ii) $\mathrm{Ext}_A^i(M,C)=0$ for all $i>0$
  (iii) the natural map $M \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Hom}_A(M,C),C)$ is an isomorphism.

(d) (i) C is a maximal CM module
  (ii) $\mathrm{injdim}_A C < \infty$
  (iii) the natural map $A \xrightarrow{\sim} \mathrm{End}_A(C) = \mathrm{Hom}_A(C,C)$ is an isomorphism.

Proof: (a) $\Rightarrow$ (b): (i) and (ii) follow from (10.34). In order to prove (iii) notice that $\mathrm{ht}\, \mathrm{ann}(M) = d-t$. Since A is CM, there is a regular sequence $\underline{x} = x_1, \dots, x_{d-t} \in \mathrm{ann}(M)$. By (10.33) $\underline{x}$ is a regular sequence on C. We claim that $C/(\underline{x})C$ is the canonical module of $A/(\underline{x})$. $C/(\underline{x})C$ is a MCM $A/(x)$-module of dimension t and by (10.14) $\mathrm{Ext}_A^d(k,C) \cong \mathrm{Ext}^t_{A/(\underline{x})}(k, C/(\underline{x})C) \cong k$ and by (10.15) $\mathrm{injdim}_{A/(\underline{x})}(C/(\underline{x})C) < \infty$. Thus $C/(\underline{x})C$ is the canonical module of $A/(\underline{x})$. By (10.28):

By (10.28) $\mathrm{Ext}_A^{d-t}(\mathrm{Ext}_A^{d-t}(M,C),C) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^{d-t}(M,C), C/(x)C)$ and $\mathrm{Ext}_A^{d-t}(M,C) \cong \mathrm{Hom}_A(M, C/(\underline{x})C)$. Thus there is a natural isomorphism $\mathrm{Ext}_A^{d-t}(\mathrm{Ext}_A^{d-t}(M,C),C) \cong \mathrm{Hom}_A(\mathrm{Hom}_A(M, C/(\underline{x})C), C/(\underline{x})C) \cong \mathrm{Hom}_{A/(\underline{x})}(\mathrm{Hom}_{A/(\underline{x})}(M/(\underline{x})M, C/(\underline{x})C), C/(\underline{x})C)$. Hence we may replace A by $A/(\underline{x})$ and assume that M is MCM A-module, i.e. $t=d$. Let $\varphi_M : M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M,C),C)$ be the natural map defined by $\varphi_M(m) : \mathrm{Hom}_A(M,C) \longrightarrow C$ with $\varphi_M(m)(f) = f(m)$. Let $\underline{y} = y_1, \ldots, y_d$ be a regular A-sequence and set $\overline{A} = A/(\underline{y})$. By (i) $\mathrm{Hom}_A(\mathrm{Hom}_A(M,C),C)$ is MCM and by (10.33) $\underline{y}$ is a regular sequence on this module. By (10.32) it suffices to show that $\varphi_M \otimes \overline{A}$ is an isomorphism. By (ii) $\mathrm{Ext}_A^i(M,C)=0$ for all $i>0$ and thus by (10.34)(c) $\mathrm{Hom}_A(M,C)\otimes\overline{A} \cong \mathrm{Hom}_{\overline{A}}(M/(\underline{y})M, C/(\underline{y})C)$. Similarly, $\mathrm{Hom}_A(M,C)$ is MCM and $\mathrm{Hom}_A(\mathrm{Hom}_A(M,C),C)\otimes_A\overline{A} \cong \mathrm{Hom}_{\overline{A}}(\mathrm{Hom}_A(M,C)\otimes\overline{A}, C/(\underline{y})C)$. Thus $\mathrm{Hom}_A(\mathrm{Hom}_A(M,C),C)\otimes\overline{A} \cong \mathrm{Hom}_{\overline{A}}(\mathrm{Hom}_{\overline{A}}(M/(\underline{y})M, C/(\underline{y})C), C/(\underline{y})C)$ and $\varphi_M\otimes\overline{A} = \varphi_{M\otimes\overline{A}}$. By (10.35) $C\otimes_A\overline{A} \cong E_{\overline{A}}(k)$. The natural map $\varphi_{M\otimes\overline{A}} : M\otimes\overline{A} \longrightarrow (M\otimes\overline{A})''$ is an isomorphism by (10.22).

(b) $\Rightarrow$ (c): trivial

(c) $\Rightarrow$ (d): (i) and (iii) follow immediately from (c) applied to $M=A$. In order to prove (ii) let N be a finitely generated A-module and let M be a (finitely generated) dth syzygy module of N. By (8.22) M is a MCM A-module since A is CM of dimension d. By assumption (c) $\mathrm{Ext}_A^1(M,C)=0$ and by (7.36) $\mathrm{Ext}_A^1(M,C) \cong \mathrm{Ext}_A^{d+1}(N,C)=0$ since M is an dth syzygy module of M. By (7.42) $\mathrm{injdim}_A C \le d < \infty$.

(d) $\Rightarrow$ (a): It remains to show that $r(C)=1$. Let $\underline{x} = x_1, \ldots, x_d$ be an A-sequence, $A/(\underline{x}) = \overline{A}$, and $E = E_{\overline{A}}(k)$. Since C is MCM, $\underline{x}$ is C-regular and $r(C) = r(C/(\underline{x})C)$ by (10.29). By (10.15) $\mathrm{injdim}_{\overline{A}}\, C/(\underline{x})C < \infty$ and $C/(\underline{x})C$ is an injective $\overline{A}$-module, hence $C/(\underline{x})C \cong E^r$ with $r = r(C/(\underline{x})C)$. Since C is MCM by (10.34)(a),(c) $A \xrightarrow{\sim} \mathrm{End}_A(C)$ specializes to an isomorphism $\overline{A} \xrightarrow{\sim} \mathrm{End}_{\overline{A}}(C/(\underline{x})C)$. But $\mathrm{End}_{\overline{A}}(C/(\underline{x})C) \cong \mathrm{Hom}_{\overline{A}}(E^r, E^r) \underset{(1)}{\cong} \mathrm{Hom}_{\overline{A}}(E,E)^{r^2} \underset{(2)}{\cong} A^{r^2}$ where (1) follows by (7.89) and (2) by (10.23). Thus $\overline{A} \cong \overline{A}^{r^2}$ and $r=1$.

(10.37) shows that $\mathrm{Ext}_A^{d-t}(-,\omega_A)$ is a contravariant functor on the category of (fin. gen) CM-modules of dimension t and defines a duality on this category; in particular, $\mathrm{Hom}_A(-,\omega_A)$ is a contravariant exact functor on the category of MCM A-modules and defines a duality on this category. Also, (10.36) and (10.37) show that among CM-rings, Gorenstein rings are exactly those rings for which $\mathrm{Hom}_A(-,A)$ is a contravariant exact functor and a duality on the category of MCM A-modules; in particular, over a local Gorenstein ring every MCM A-module is reflexive (i.e. the natural map $M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A)$ is an isomorphism).

Recall: If (A,m,k) is a local Noetherian ring and M a finitely generated A-module, then $\mu(M) = \dim_k(k\otimes_A M)$ is the minimal number of generators of M.

(10.38) Proposition: Let (A,m,k) be a local CM-ring of dimension d with a canonical module ω_A.
(a) Let M be a CM A-module of dimension t; then $\mu(\mathrm{Ext}_A^{d-t}(M,\omega_A)) = r(M)$ and $r(\mathrm{Ext}_A^{d-t}(M,\omega_A)) = \mu(M)$.
(b) ω_A is a faithful A-module with $\mu(\omega_A) = r(A)$ and $r(\omega_A) = 1$.

Proof: (a) As in the proof of (10.37) we can reduce to the case where $\dim A = 0$. Then $\omega_A \cong E_A(k) = E$ and by (10.22) $\mathrm{Hom}_A(M,E) = M'$ and $M'' = \mathrm{Hom}_A(\mathrm{Hom}_A(M,E),E) \cong M$. By (6.42) $\mathrm{Soc}(M') \cong \mathrm{Hom}_A(k,\mathrm{Hom}_A(M,E)) \cong \mathrm{Hom}_A(k\otimes_A M, E)$ and $r(M') = \ell_A(k\otimes_A M) = \mu(M)$. By (10.22) $r(M) = r((M')') = \mu(M')$.
(b) By (10.37) $\mathrm{End}_A(\omega_A) \cong A$ and ω_A is faithful. The rest follows from (a) with $M = A$

(10.39) Theorem: Let A be a local CM-ring with a canonical module ω_A and $P \in \mathrm{Spec}(A)$. Then A_P has a canonical module and $\omega_{A_P} \cong (\omega_A)_P$.

Proof: The conditions of (10.37)(d) are preserved under localization.

(10.40) Lemma: Let (A, m, k) be a local CM-ring of dimension d and C a finitely generated A-module. The following are equivalent:

(a) C is a canonical module of A

(b) $\mu_i(m, C) = \delta_{id}$ for all i.

Proof: By (7.66) $\mu_i(m,C) = \dim_k \operatorname{Ext}^i_A(k, C)$ and by (8.18) C is MCM if and only $\mu_i(m,C) = 0$ for $i<d$ and $\mu_d(m,C) \neq 0$. In this case $r(C) = \mu_d(m, C)$. Moreover, $\operatorname{injdim}_A C < \infty$ if and only if $\operatorname{injdim}_A C \leq d$ which is equivalent to $\mu_i(m,C) = 0$ for $i > d$ by (10.12).

(10.41) Theorem: Let A be a local CM-ring and C a finitely generated A-module. The following are equivalent:

(a) $C \cong \omega_A$

(b) $\mu_i(P, C) = \delta_{i\,\mathrm{ht} P}$ for all $i \geq 0$ and all $P \in \operatorname{Spec}(A)$

(c) Let $I^\bullet$ be a minimal injective A-resolution of C. Then $I^i = \bigoplus E_A(A/P)$, where P runs over all prime ideals of height i for all $i \geq 0$.

Proof: (a)$\Longleftrightarrow$(b): By (10.39) $C \cong \omega_A$ if and only if $C_P \cong \omega_{A_P}$ for all $P \in \operatorname{Spec}(A)$. By (10.40) this is equivalent to $\mu_i(PA_P, C_P) = \delta_{i \dim A_P} = \delta_{i\,\mathrm{ht} P} = \mu_i(P, C)$.

(b)$\Longleftrightarrow$(c): Use (7.68)

(10.42) Proposition: Let A be a local CM-ring and C a finitely generated A-module.

(a) Let $\underline{x}$ be a regular sequence on A and C. Then $C \cong \omega_A$ if and only if $C/(\underline{x})C \cong \omega_{A/(\underline{x})}$.

(b) $C \cong \omega_A$ if and only if $\hat{C} \cong \omega_{\hat{A}}$.

Proof: (a) Homework

(b) By (7.90) $\operatorname{Ext}^i_{\hat{A}}(\hat{A}/\hat{m}, \hat{C}) \cong \operatorname{Ext}^i_A(A/m, C) \otimes_A \hat{A}$. Thus $\mu_i(\hat{m}, \hat{C}) = \mu_i(m, C)$ and the assertion follows by (10.40).

(10.43) Theorem: Let $\varphi : A \longrightarrow B$ be a local homomorphism of local CM-rings so that B is a finitely generated A-module and let $g = \dim A - \dim B$. If A has a canonical module, then so does B and $\omega_B \cong \operatorname{Ext}_A^g(B, \omega_A)$.

Proof: Set $I = \ker\varphi$. Then $A/I \hookrightarrow B$ is an integral extension and $\dim A/I = \dim B$. Since A is CM, by (8.33) $\dim A = \operatorname{ht} I + \dim A/I$ and $g = \dim A - \dim A/I = \operatorname{ht} I$. Let $\underline{x} = x_1, \dots, x_g \in I$ be an A-regular sequence. By (10.33) $\underline{x}$ is a regular sequence on ω_A and by (10.14) $\operatorname{Ext}_A^g(B, \omega_A) \cong \operatorname{Hom}_{A/(\underline{x})}(B, \omega_A/(\underline{x})\omega_A)$. Moreover, by (10.42) $\omega_A/(\underline{x})\omega_A \cong \omega_{A/(\underline{x})}$. $A/(\underline{x})$ is a CM-ring of dimension $\dim A - g = \dim B$. Thus we may replace A by $A/(\underline{x})$ and assume that $\dim A = d = \dim B$. We have to show that $\operatorname{Hom}_A(B, \omega_A)$ is the canonical module of B.

Let $\underline{y} = y_1, \dots, y_d$ be a SoP of A. Then $\underline{y}$ is a SoP of B, since φ is local, finite, and $\dim B = d$. Since B is CM, $\underline{y}$ is a regular sequence on B. Thus B is a MCM A-module. By (10.34) $\operatorname{Hom}_A(B, \omega_A)$ is a MCM A-module and by (10.33) $\underline{y}$ is a regular sequence on $\operatorname{Hom}_A(B, \omega_A)$. By (10.42)(a) it suffices to show that $\omega_{\overline{B}} \cong \operatorname{Hom}_A(B, \omega_A) \otimes_B \overline{B}$, where $\overline{B} = B/(\underline{y})$. Set $\overline{A} = A/(\underline{y})$. By (10.34) $\operatorname{Hom}_A(B, \omega_A) \otimes_B \overline{B} \cong \operatorname{Hom}_A(B, \omega_A) \otimes_A \overline{A} \cong \operatorname{Hom}_{\overline{A}}(B \otimes_A \overline{A}, \omega_A \otimes_A \overline{A}) \cong \operatorname{Hom}_{\overline{A}}(\overline{B}, \omega_{\overline{A}})$. Thus we may replace A, B by $\overline{A}, \overline{B}$ and may assume that $\dim A = \dim B = 0$.

Let k, ℓ be the residue fields of A and B. Then $\omega_A = E_A(k)$ and we have to show that $E_B(\ell) \cong \operatorname{Hom}_A(B, \omega_A)$ as B-modules. There is an adjoint isomorphism $\operatorname{Hom}_A(M \otimes_B B, \omega_A) \cong \operatorname{Hom}_B(M, \operatorname{Hom}_A(B, \omega_A))$ for all B-modules M (compare with (6.42)). Since ω_A is an injective A-module, $\operatorname{Hom}_A(-, \omega_A)$ is exact. Thus $\operatorname{Hom}_B(-, \operatorname{Hom}_A(B, \omega_A))$ is exact and $\operatorname{Hom}_A(B, \omega_A)$ is an injective B-module. Thus $\operatorname{Hom}_A(B, \omega_A) \cong E_B(\ell)^r$. By (10.22) $\ell_A(\operatorname{Hom}_A(B, \omega_A)) = \ell_A(B)$ and $\ell_A(E_B(\ell)^r) = r \cdot \ell_B(E_B(\ell)) \cdot \dim_k \ell = r \cdot \ell_B(B) \dim_k \ell = r \cdot \ell_A(B)$. Thus $r = 1$.

(10.44) <u>Corollary</u>: Every complete local CM-ring has a canonical module.

<u>Proof</u>: By (9.40) every complete local ring is factor ring of a RLR. Use (10.43).

(10.45) <u>Examples</u>: Let B be a local CM-ring.

(a) Assume $B \cong A/I$ with A a local Gorenstein ring and I an A-ideal of grade g. Let $\underline{x} = x_1, \ldots, x_g \in I$ be an A-regular sequence. Then $\mathrm{Ext}_A^g(B, A) \cong \mathrm{Hom}_{A/(\underline{x})}(A/I, A/(\underline{x})) \cong \omega_B$.

(b) Assume that B is complete and contains a field: Let k be a coefficient field of B; $x_1, \ldots, x_d$ a SOP of B, and write $A = k[[x_1, \ldots, x_d]] \subseteq B$. Then A is a power series ring and B is finite over A. By (10.43) $\omega_B \cong \mathrm{Hom}_A(B, A)$.

Let A be a ring and M an A-module. We construct a ring extension $A \subseteq A * M$ of A, called the <u>trivial extension</u> of A by M as follows: As an A-module $A * M = A \oplus M$ and multiplication is defined by $(a,x)(b,y) = (ab, ay + bx)$ for all $a, b \in A$ and $x, y \in M$. Note that $M \subseteq A * M$ is an ideal with $M^2 = 0$ and $A * M/M \cong A$.

(10.46) <u>Theorem</u>: Let A be a local CM-ring. Then A has a canonical module if and only if A is a factor ring of a local Gorenstein ring.

<u>Proof</u>: "$\Leftarrow$": Use (10.43)

"$\Rightarrow$": It is enough to show that $B = A * \omega_A$ is a local Gorenstein ring. Let $d = \dim A$. Since $A \subseteq B$ is a finite ring extension, B is a Noetherian ring with $\dim B = d$. Since $\omega_A^2 = 0$ in B and $B/\omega_A \cong A$ local, the ring B is local. Let $\underline{x} = x_1, \ldots, x_d$ be an A-sequence. Then $\underline{x}$ is regular on ω_A and hence on $B = A * \omega_A$. Thus B is CM. It remains to show that $r(B) = 1$, or equivalently, $r(B/(\underline{x})B) = 1$. Since $B/(\underline{x})B \cong A/(\underline{x}) * \omega_{A/(\underline{x})}$ we may replace A by $A/(\underline{x})$ and assume that $\dim A = 0$. In this case $\omega_A = E_A(k)$. It remains to show that $r(B) = r(A * E_A(k)) = 1$.

Let $(a,x) \in \mathcal{J}(B)$. Then for all $b \in m$: $(b,0)(a,x) = (ab, bx) = (0,0)$ and $a \in \mathcal{J}(A)$ and $x \in \mathcal{J}(E_A(k))$. If $a \neq 0$, the exact sequence $A \xrightarrow{a} A \longrightarrow A/(a) \longrightarrow 0$ induces an exact sequence $0 \longrightarrow \mathrm{Hom}_{A/(a)}(A/(a), E_A(k)) \longrightarrow \mathrm{Hom}_A(A, E_A(k)) \xrightarrow{a} \mathrm{Hom}_A(A, E_A(k))$. By a similar argument as in the proof of (10.43) (via the adjoint isomorphism) $\mathrm{Hom}_{A/(a)}(A/(a), E_A(k))$ is an injective $A/(a)$-module. Since $\mathcal{J}(\mathrm{Hom}_{A/(a)}(A/(a), E_A(k))) \cong k$, we have that $\mathrm{Hom}_{A/(a)}(A/(a), E_A(k)) \cong E_{A/(a)}(k)$ and the sequence $0 \longrightarrow E_{A/(a)}(k) \longrightarrow E_A(k) \xrightarrow{a} E_A(k)$ is exact. Moreover, $\ell(E_{A/(a)}(k)) = \ell(A/(a)) < \ell(A) = \ell(E_A(k))$ and multiplication by a on $E_A(k)$ cannot be the zero map. Thus there is a $y \in E_A(k)$ with $ay \neq 0$ and $(0,y)(a,x) = (0, ay) \neq (0,0)$, a contradiction. Thus $\mathcal{J}(A * E_A(k)) \cong \mathcal{J}(E_A(k))$ and $r(A * E_A(k)) = 1$.