§5: DIRECT LIMITS

(7.69) Definition: (a) A *directed set* is a partially ordered set Λ such that for all $\lambda, \mu \in \Lambda$ there is an element $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$.

(b) Let A be a ring, Λ a directed set, and $\{M_\lambda \mid \lambda \in \Lambda\}$ a family of A-modules. Suppose that for each pair $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ there is given an A-linear map $f_{\mu\lambda}: M_\lambda \longrightarrow M_\mu$ so that (i) $f_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ for all $\lambda \in \Lambda$ and (ii) $f_{\nu\mu} f_{\mu\lambda} = f_{\nu\lambda}$ whenever $\lambda \leq \mu \leq \nu$. Then $\{M_\lambda, f_{\mu\lambda}\}$ is called a *direct system* (of A-modules) over the *directed* set Λ.

Note that one can define direct systems of sets, groups, rings, etc. accordingly.

(7.70) Definition: (a) Let $\{M_\lambda, f_{\mu\lambda}\}$ and $\{M'_\lambda, f'_{\mu\lambda}\}$ be direct systems of A-modules over the directed set Λ. A *morphism* (or an A-*linear map*) $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \{M'_\lambda, f'_{\mu\lambda}\}$ is a family of A-linear maps $\varphi = \{\varphi_\lambda : M_\lambda \longrightarrow M'_\lambda \mid \lambda \in \Lambda \text{ and } \varphi_\lambda \text{ A-linear}\}$ so that for all $\lambda \leq \mu$, $f'_{\mu\lambda} \circ \varphi_\lambda = \varphi_\mu \circ f_{\mu\lambda}$, that is, the diagram:

$$\begin{tikzcd}
M_\lambda \arrow[r, "f_{\mu\lambda}"'] \arrow[d, "\varphi_\lambda"'] & M_\mu \arrow[d, "\varphi_\mu"] \\
M'_\lambda \arrow[r, "f'_{\mu\lambda}"'] & M'_\mu
\end{tikzcd}$$

commutes.

(b) Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of A-modules over Λ and X an A-module. A *morphism* (or A-linear map) $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow X$ is a family of A-linear maps $\varphi = \{\varphi_\lambda : M_\lambda \longrightarrow X \mid \lambda \in \Lambda \text{ and } \varphi_\lambda \text{ A-linear}\}$ so that for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, $\varphi_\lambda = \varphi_\mu \circ f_{\mu\lambda}$, that is, the diagram:

$$\begin{tikzcd}
M_\lambda \arrow[r, "\varphi_\lambda"] \arrow[d, "f_{\mu\lambda}"'] & X \\
M_\mu \arrow[ur, "\varphi_\mu"'] &
\end{tikzcd}$$

commutes.

(c) Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of A-modules over Λ. An A-module M_∞ together with an A-linear map $\psi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow M_\infty$ is called a *direct limit* of $\{M_\lambda, f_{\mu\lambda}\}$ if for every A-linear map $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow X$ there is a unique A-linear map $h: M_\infty \longrightarrow X$ so that $\varphi_\lambda = h\psi_\lambda$ for all $\lambda \in \Lambda$, that is, the diagram:

$$\begin{tikzcd}
M_\lambda \arrow[d, "f_{\mu\lambda}"'] \arrow[dr, "\psi_\lambda"] \arrow[drr, bend left, "\varphi_\lambda"] & & \\
& M_\infty \arrow[r, "h"] & X \\
M_\mu \arrow[ur, "\psi_\mu"'] \arrow[urr, bend right, "\varphi_\mu"'] & &
\end{tikzcd}$$

<u>Notation</u>: $M_\infty = \varinjlim M_\lambda = \varinjlim_{\Lambda} M_\lambda$

(7.71) <u>Theorem</u>: Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of A-modules over Λ. Then $\varinjlim M_\lambda$ exists.

<u>Proof</u>: Let $C = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be the direct sum of the M_λ and $i_\lambda : M_\lambda \longrightarrow \bigoplus_{\mu\in\Lambda} M_\mu$ the natural embedding. Let $D \subseteq C$ be the submodule which is generated by:

$$\{\, i_\lambda(a) - i_\mu(f_{\mu\lambda}(a)) \mid a \in M_\lambda\,;\ \lambda \le \mu \,\}.$$

Set $M_\infty = C/D$ and define $\psi : \{M_\lambda, f_{\mu\lambda}\} \longrightarrow M_\infty$ by $\psi_\lambda(a) = i_\lambda(a) + D$. Obviously, $\psi_\lambda = \psi_\mu f_{\mu\lambda}$ for $\lambda \le \mu$.

Let $\varphi : \{M_\lambda, f_{\mu\lambda}\} \longrightarrow X$ be an A-linear map. By the universal property of the direct sum there is a unique A-linear map $g : C = \bigoplus M_\lambda \longrightarrow X$ so that $g i_\lambda = \varphi_\lambda$ for all $\lambda \in \Lambda$. Then for all $\lambda \in \Lambda$ and all $a \in M_\lambda$:

$$g\big(i_\lambda(a) - i_\mu(f_{\mu\lambda}(a))\big) = (g i_\lambda)(a) - (g i_\mu)(f_{\mu\lambda}(a)) = \varphi_\lambda(a) - \varphi_\mu f_{\mu\lambda}(a) = 0.$$

g factors through M_∞:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "\nu"'] & X \\
M_\infty \arrow[ur, "h"'] &
\end{tikzcd}$$

where ν is the canonical map. Moreover, for all $\lambda \in \Lambda$ the diagram

$$\begin{tikzcd}
M_\lambda \arrow[r, "i_\lambda"'] \arrow[rr, bend left, "\psi_\lambda"] \arrow[dr, "\varphi_\lambda"'] & C \arrow[r, "\nu"'] \arrow[d, "g"] & M_\infty \arrow[dl, "h"] \\
& X &
\end{tikzcd}$$

commutes.

The uniqueness of h follows since M_∞ is generated by $\bigcup_{\lambda\in\Lambda} \psi_\lambda(M_\lambda)$. Thus $M_\infty = \varinjlim M_\lambda$ together with ψ is a direct limit of $\{M_\lambda, f_{\mu\lambda}\}$.

(7.72) <u>Proposition</u>: The direct limit $\varinjlim M_\lambda$ of a direct system $\{M_\lambda, f_{\mu\lambda}\}$ is unique up to isomorphism.

Proof: Homework

(7.73) Examples: (a) Let M be an A-module, $M_\lambda \subseteq M$ submodules, $\lambda \leq \mu$ if and only if $M_\lambda \subseteq M_\mu$, and $f_{\mu\lambda}: M_\lambda \longrightarrow M_\mu$ the inclusion. Then $\varinjlim M_\lambda \cong \bigcup_{\lambda\in\Lambda} M_\lambda$.
(b) Let M be an A-module and $\{M_\lambda, \text{incl.}\}$ the direct system of all finitely generated submodules of M. Then $M \cong \varinjlim M_\lambda$ is a direct limit of finitely generated A-modules.
(c) Let A be a ring and $\{A_\lambda, \text{incl.}\}$ the direct system of all finitely generated $\mathbb{Z}$-subalgebras of A. Then $A \cong \varinjlim A_\lambda$ is a direct limit of Noetherian rings.

(7.74) Remark: Let $\{M_\lambda, f_{\mu\lambda}\}$ and $\{M'_\lambda, f'_{\mu\lambda}\}$ be direct systems of A-modules over the directed set Λ and let $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \{M'_\lambda, f'_{\mu\lambda}\}$ be an A-linear map. Then there is a unique A-linear map $\varinjlim \varphi_\lambda : \varinjlim M_\lambda \longrightarrow \varinjlim M'_\lambda$ with $\psi'_\lambda \varphi_\lambda = \varinjlim \varphi_\lambda \psi_\lambda$.
(where $\psi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \varinjlim M_\lambda$, $\psi': \{M'_\lambda, f'_{\mu\lambda}\} \longrightarrow \varinjlim M'_\lambda$ are the natural maps.)

Proof: By the definition of the direct limit (7.70)(c).

(7.75) Proposition: Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system with direct limit $\varinjlim M_\lambda$ and natural maps $\psi_\lambda: M_\lambda \longrightarrow \varinjlim M_\lambda$.
(a) For all $m \in \varinjlim M_\lambda$ there is a $\lambda \in \Lambda$ and an element $m_\lambda \in M_\lambda$ with $\psi_\lambda(m_\lambda) = m$.
(b) If $m_\lambda \in M$ with $\psi_\lambda(m_\lambda) = 0$ then there is a $\mu \geq \lambda$ with $f_{\mu\lambda}(m_\lambda) = 0$.

Proof: (a) Let $m \in \varinjlim M_\lambda$. Since $\varinjlim M_\lambda$ is generated by $\{\psi_\lambda(M_\lambda)\}_{\lambda\in\Lambda}$ we can write $m = \sum_{i=1}^{r} \psi_{\tau_i}(m_{\tau_i})$. Let $\lambda \in \Lambda$ with $\tau_i \leq \lambda$ for all $1 \leq i \leq r$. Then $\psi_{\tau_i}(m_{\tau_i}) = \psi_\lambda(f_{\lambda\tau_i}(m_{\tau_i}))$ and with $m_\lambda = \sum f_{\lambda\tau_i}(m_{\tau_i})$ we have that $m = \psi_\lambda(m_\lambda)$.
(b) Let $i_\lambda: M_\lambda \longrightarrow C = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be the canonical injections, $D \subseteq C$ the submodule generated by $\{ i_\lambda(a) - i_\mu(f_{\mu\lambda}(a)) \mid a \in M_\lambda;\ \lambda \leq \mu\}$. By construction of $\varinjlim M_\lambda = C/D$, $\psi_\lambda(m_\lambda) = 0$ implies that:

$$(*) \quad i_\lambda(m_\lambda) = \sum_{\tau} \left(i_{\mu(\tau)} f_{\mu(\tau)\tau}(m_\tau) - i_\tau(m_\tau) \right)$$

Let $\sigma \in \Lambda$ be larger than any index which occurs in $(*)$. In the following we write μ instead of $\mu(\tau)$ with μ dependent on τ. Then

$$\begin{aligned} i_\sigma f_{\sigma\lambda}(m_\lambda) &= (i_\sigma f_{\sigma\lambda}(m_\lambda) - i_\lambda(m_\lambda)) + i_\lambda(m_\lambda) \\ &= (i_\sigma f_{\sigma\lambda}(m_\lambda) - i_\lambda(m_\lambda)) + \sum_\tau (i_\mu f_{\mu\tau}(m_\tau) - i_\tau(m_\tau)) \end{aligned}$$

For each τ:

$$\begin{aligned} i_\mu f_{\mu\tau}(m_\tau) - i_\tau(m_\tau) = &(i_\sigma f_{\sigma\tau}(m_\tau) - i_\tau(m_\tau)) + \\ &+ [i_\sigma f_{\sigma\mu}(-f_{\mu\tau}(m_\tau)) - i_\mu(-f_{\mu\tau}(m_\tau))] \end{aligned}$$

Thus we can write:

$$i_\sigma f_{\sigma\lambda}(m_\lambda) = \sum_\tau (i_\sigma f_{\sigma\tau}(n_\tau) - i_\tau(n_\tau)) \in \bigoplus M_\lambda$$

where σ is independent of τ. If $\tau \neq \sigma$, then $i_\tau(n_\tau) = 0$ and $n_\tau = 0$. This implies:

$$i_\sigma f_{\sigma\lambda}(m_\lambda) = i_\sigma f_{\sigma\sigma}(n_\sigma) - i_\sigma(n_\sigma) = 0$$

since $f_{\sigma\sigma} = \mathrm{id}$. Therefore $f_{\sigma\lambda}(m_\lambda) = 0$.

(7.76) <u>Theorem</u>: Let $\{M'_\lambda, f'_{\mu\lambda}\}$, $\{M_\lambda, f_{\mu\lambda}\}$, $\{M''_\lambda, f''_{\mu\lambda}\}$ be direct systems of A-modules over the directed set Λ and let $\{\alpha_\lambda\}: \{M'_\lambda, f'_{\mu\lambda}\} \longrightarrow \{M_\lambda, f_{\mu\lambda}\}$ and $\{\beta_\lambda\}: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \{M''_\lambda, f''_{\mu\lambda}\}$ be A-linear maps so that for all $\lambda \in \Lambda$ the sequence $M'_\lambda \xrightarrow{\alpha_\lambda} M_\lambda \xrightarrow{\beta_\lambda} M''_\lambda$ is exact. Then the sequence:

$$\varinjlim M'_\lambda \xrightarrow{\varinjlim \alpha_\lambda} \varinjlim M_\lambda \xrightarrow{\varinjlim \beta_\lambda} \varinjlim M''_\lambda \quad \text{is exact.}$$

<u>Proof</u>: Let $\psi_\lambda: M_\lambda \longrightarrow \varinjlim M_\lambda$ and $\psi'_\lambda, \psi''_\lambda$, respectively, denote the canonical maps into the direct limits. Set $\alpha_\infty = \varinjlim \alpha_\lambda$ and $\beta_\infty = \varinjlim \beta_\lambda$. For all $\lambda \in \Lambda$ we have commutative diagrams:

$$\begin{tikzcd} M'_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d, "\psi'_\lambda"'] & M_\lambda \arrow[d, "\psi_\lambda"] \\ \varinjlim M'_\lambda \arrow[r, "\alpha_\infty"] & \varinjlim M_\lambda \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M_\lambda \arrow[r, "\beta_\lambda"] \arrow[d, "\psi_\lambda"'] & M''_\lambda \arrow[d, "\psi''_\lambda"] \\ \varinjlim M_\lambda \arrow[r, "\beta_\infty"] & \varinjlim M''_\lambda \end{tikzcd}$$

(1) $\beta_\infty \alpha_\infty = 0$.

<u>Pf(1)</u>: Let $z' \in \varinjlim M'_\lambda$. Then there is a $\lambda \in \Lambda$ and an $m'_\lambda \in M'_\lambda$ with $\psi'_\lambda(m'_\lambda) = z'$. Thus $\alpha_\infty(z') = \alpha_\infty \psi'_\lambda(m'_\lambda) = \psi_\lambda \alpha_\lambda(m'_\lambda)$ and $\beta_\infty \alpha_\infty(z') = \beta_\infty \psi_\lambda \alpha_\lambda(m'_\lambda) = \psi''_\lambda \beta_\lambda \alpha_\lambda(m'_\lambda) = 0$.

(2) $\ker(\beta_\infty) \subseteq \operatorname{im}(\alpha_\infty)$

Pf (2): Let $z \in \varinjlim M_\lambda$ with $\beta_\infty(z) = 0$ and let $m_\lambda \in M_\lambda$ with $\psi_\lambda(m_\lambda) = z$. Then $\beta_\infty \psi_\lambda(m_\lambda) = \psi''_\lambda \beta_\lambda(m_\lambda) = 0$. By (7.75) there is a $\mu \in \Lambda$ with $\lambda \leq \mu$ so that $f''_{\mu\lambda}\beta_\lambda(m_\lambda) = 0$. Since $f''_{\mu\lambda}\beta_\lambda = \beta_\mu f_{\mu\lambda}$, we obtain $\beta_\mu(f_{\mu\lambda}(m_\lambda)) = 0$ and therefore $f_{\mu\lambda}(m_\lambda) \in \ker(\beta_\mu) = \operatorname{im}(\alpha_\mu)$. Let $m'_\mu \in M'_\mu$ with $\alpha_\mu(m'_\mu) = f_{\mu\lambda}(m_\lambda)$. Then $\alpha_\infty \psi'_\mu(m'_\mu) = \psi_\mu \alpha_\mu(m'_\mu) = \psi_\mu f_{\mu\lambda}(m_\lambda) = \psi_\lambda(m_\lambda) = z$.

(7.77) Theorem: Let A be a ring, N an A-module, and $\{M_\lambda, f_{\mu\lambda}\}$ a direct system of A-modules over Λ. Then $\{M_\lambda \otimes_A N, f_{\mu\lambda} \otimes \mathrm{id}\}$ is a direct system of A-modules over Λ and

$$\varinjlim (M_\lambda \otimes_A N) \cong (\varinjlim M_\lambda) \otimes_A N$$

i.e. the tensor product commutes with direct limits.

Proof: Obviously, $\{M_\lambda \otimes_A N, f_{\mu\lambda} \otimes \mathrm{id}\}$ is a direct system of A-modules. There is an isomorphism $\bigoplus (M_\lambda \otimes_A N) \overset{(*)}{\cong} (\bigoplus M_\lambda) \otimes_A N$ which maps $i_\lambda(a \otimes b)$ to $i_\lambda(a) \otimes b$. Under this map, $i_\lambda(a \otimes b) - i_\mu((f_{\mu\lambda} \otimes \mathrm{id})(a \otimes b))$ is mapped to $(i_\lambda(a) - i_\mu(f_{\mu\lambda}(a))) \otimes b$. Let D be the submodule of $\bigoplus M_\lambda$ generated by $\{ i_\lambda(a) - i_\mu(f_{\mu\lambda}(a)) \mid a \in M_\lambda, \lambda \leq \mu \}$. The sequence

$$D \otimes_A N \xrightarrow{h} (\bigoplus M_\lambda) \otimes_A N \longrightarrow (\bigoplus M_\lambda / D) \otimes_A N \longrightarrow 0$$

is exact and under the isomorphism $(*)$, the submodule $\tilde{D}$ of $\bigoplus (M_\lambda \otimes_A N)$ which is generated by $\{ i_\lambda(a \otimes b) - i_\mu(f_{\mu\lambda} \otimes \mathrm{id})(a \otimes b) \mid a \otimes b \in M_\lambda \otimes N, \lambda \leq \mu \}$ is mapped onto $\operatorname{im}(h)$. This shows the assertion.

§6: INVERSE SYSTEMS

(7.78) Definition: Let A be a ring, Λ a directed set, and $\{M_\lambda, \lambda\in\Lambda\}$ a family of A-modules. Suppose that for each pair $\lambda,\mu\in\Lambda$ with $\lambda\leq\mu$ there is given an A-linear map: $f_{\lambda\mu}: M_\mu \longrightarrow M_\lambda$ so that (i) $f_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ for all $\lambda\in\Lambda$ and (ii) $f_{\lambda\mu} f_{\mu\nu} = f_{\lambda\nu}$ whenever $\lambda\leq\mu\leq\mu$. Then $\{M_\lambda, f_{\lambda\mu}\}$ is called an inverse system of A-modules over Λ. Inverse systems of sets, groups, rings etc. are defined accordingly.

(7.79) Definition: (a) Let $\{M_\lambda, f_{\lambda\mu}\}$ and $\{M'_\lambda, f'_{\lambda\mu}\}$ be inverse systems of A-modules over Λ. A morphism (or an A-linear map) $\varphi: \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M'_\lambda, f'_{\lambda\mu}\}$ is a family of A-linear maps $\varphi = \{\varphi_\lambda: M_\lambda \longrightarrow M'_\lambda \mid \lambda\in\Lambda \text{ and } \varphi_\lambda \text{ } A\text{-linear}\}$ so that for all $\lambda\leq\mu$: $f'_{\lambda\mu}\varphi_\mu = \varphi_\lambda f_{\lambda\mu}$, that is, the diagram:

$$\begin{tikzcd} M_\mu \arrow[r, "\varphi_\mu"] \arrow[d, "f_{\lambda\mu}"'] & M'_\mu \arrow[d, "f'_{\lambda\mu}"] \\ M_\lambda \arrow[r, "\varphi_\lambda"] & M'_\lambda \end{tikzcd}$$

commutes.

(b) Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of A-modules over Λ and X an A-module. A morphism (or an A-linear map) $\varphi: X \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ is a family of A-linear maps $\varphi = \{\varphi_\lambda: X \longrightarrow M_\lambda \mid \lambda\in\Lambda \text{ and } \varphi_\lambda \text{ } A\text{-linear}\}$ so that for all $\lambda,\mu\in\Lambda$ with $\lambda\leq\mu$, $\varphi_\lambda = f_{\lambda\mu}\varphi_\mu$, that is, the diagram:

$$\begin{tikzcd} X \arrow[r, "\varphi_\mu"] \arrow[dr, "\varphi_\lambda"'] & M_\mu \arrow[d, "f_{\lambda\mu}"] \\ & M_\lambda \end{tikzcd}$$

commutes.

(c) Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of A-modules over Λ. An A-module M_∞ together with an A-linear map $\psi: M_\infty \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ is called an inverse limit of $\{M_\lambda, f_{\lambda\mu}\}$ if for every A-linear map $\varphi: X \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ there is a unique A-linear map $h: X \longrightarrow M_\infty$ so that $\varphi_\lambda = \psi_\lambda h$ for all $\lambda\in\Lambda$, that is, for all $\lambda,\mu\in\Lambda$ with $\lambda\leq\mu$ the diagram

$$\begin{tikzcd} & & M_\mu \arrow[dd, "f_{\lambda\mu}"] \\ X \arrow[r, "h"] \arrow[urr, bend left, "\varphi_\mu"] \arrow[drr, bend right, "\varphi_\lambda"'] & M_\infty \arrow[ur, "\psi_\mu"] \arrow[dr, "\psi_\lambda"'] & \\ & & M_\lambda \end{tikzcd}$$

commutes.

Notation: $M_\infty = \varprojlim M_\lambda = \varprojlim_{\Lambda} M_\lambda$.

(7.80) Theorem: Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of A-modules over Λ. The inverse limit $\varprojlim M_\lambda$ exists.

Proof: For all $\lambda \in \Lambda$ let $p_\lambda : \prod M_\lambda \longrightarrow M_\lambda$ be the λ-th projection. Set

$$M_\infty = \{ x \in \prod M_\lambda \mid p_\lambda(x) = f_{\lambda\mu}(p_\mu(x)) \text{ whenever } \lambda \le \mu \}.$$

M_∞ is a submodule of $\prod M_\lambda$. For all $\lambda \in \Lambda$ let $\psi_\lambda : M_\infty \longrightarrow M_\lambda$ be the restriction of p_λ to M_∞. For all $\lambda, \mu \in \Lambda$ with $\lambda \le \mu$ and all $x \in M_\infty$: $\psi_\lambda(x) = p_\lambda(x) = f_{\lambda\mu}(p_\mu(x)) = f_{\lambda\mu}(\psi_\mu(x))$. Thus $\psi_\lambda = f_{\lambda\mu}\psi_\mu$ and $\{\psi_\lambda\} : M_\infty \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ is an A-linear map. We claim that M_∞ together with $\{\psi_\lambda\}$ is an inverse limit of $\{M_\lambda, f_{\lambda\mu}\}$. Let $\varphi = \{\varphi_\lambda\} : X \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ be an A-linear map, that is, $f_{\lambda\mu}\varphi_\mu = \varphi_\lambda$ whenever $\lambda \le \mu$. By the universal property of the direct product there is an A-linear map $g : X \longrightarrow \prod M_\lambda$ with $p_\lambda g = \varphi_\lambda$ for all $\lambda \in \Lambda$. If $\lambda \le \mu$, then

$$p_\lambda(g(x)) = \varphi_\lambda(x) = f_{\lambda\mu}\varphi_\mu(x) = f_{\lambda\mu}(p_\mu(g(x)))$$

and $g(x) \in M_\infty$. Thus g induces an A-linear map $h : X \longrightarrow M_\infty$ with $\psi_\lambda h = \varphi_\lambda$ for all $\lambda \in \Lambda$. The uniqueness of g yields the uniqueness of h.

Note: The inverse limit $\varprojlim M_\lambda$ is uniquely determined up to isomorphism.

(7.81) Example: Let $p \in \mathbb{Z}$ be a prime number. For all $n \in \mathbb{N}$, $n \ge 1$, let $f_{n+k,n} : \mathbb{Z}/(p^{n+k}) \longrightarrow \mathbb{Z}/(p^n)$ be the canonical surjection. The set $\{\mathbb{Z}/(p^n), f_{n+k,n}\}$ is an inverse system. Its inverse limit $\varprojlim \mathbb{Z}/(p^n) = \mathbb{Z}_p$ is the ring of p-adic numbers (integers).

(7.82) Remark: Let $\varphi = \{\varphi_\lambda\} : \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M'_\lambda, f'_{\lambda\mu}\}$ be an A-linear map of inverse systems. Suppose that $\psi_\lambda : \varprojlim M_\lambda \longrightarrow M_\lambda$ and $\psi'_\lambda : \varprojlim M'_\lambda \longrightarrow M'_\lambda$ are the structure maps of the inverse limits. For all $\lambda, \mu \in \Lambda$ with $\lambda \le \mu$ we have a commutative diagram:

$$\begin{tikzcd} & M_\mu \arrow[r, "\varphi_\mu"] \arrow[d, "f_{\lambda\mu}"] & M'_\mu \arrow[d, "f'_{\lambda\mu}"] \\ \varprojlim M_\lambda \arrow[ur, "\psi_\mu"] \arrow[dr, "\psi_\lambda"'] & & \\ & M_\lambda \arrow[r] & M'_\lambda \end{tikzcd}$$

This induces an A-linear map $\varprojlim \varphi_\lambda : \varprojlim M_\lambda \longrightarrow \varprojlim M'_\lambda$.

There is a different way to think of $\varprojlim \varphi_\lambda$. We have that $\varprojlim M_\lambda \subseteq \prod M_\lambda$ and $\varprojlim M'_\lambda \subseteq \prod M'_\lambda$. $\varphi = \{\varphi_\lambda\}$ induces an A-linear map $\prod \varphi_\lambda$ on the direct products:

$$\begin{array}{ccc} \prod M_\lambda & \xrightarrow{\prod \varphi_\lambda} & \prod M_\lambda' \\ \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ \varprojlim M_\lambda & \xrightarrow{\varprojlim \varphi_\lambda} & \varprojlim M'_\lambda \end{array}$$

Then $\varprojlim \varphi_\lambda = \prod \varphi_\lambda \big|_{\varprojlim M_\lambda}$.

(7.83) Theorem: Let $\{M_\lambda, f_{\lambda\mu}\}$, $\{M'_\lambda, f'_{\lambda\mu}\}$, $\{M''_\lambda, f''_{\lambda\mu}\}$ be inverse systems of A-modules over $\Lambda = \mathbb{N}$. Suppose that $\alpha = \{\alpha_\lambda\} : \{M'_\lambda, f'_{\lambda\mu}\} \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ and $\beta = \{\beta_\lambda\} : \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M''_\lambda, f''_{\lambda\mu}\}$ are A-linear maps of inverse systems with $0 \longrightarrow M'_\lambda \xrightarrow{\alpha_\lambda} M_\lambda \xrightarrow{\beta_\lambda} M''_\lambda$ exact for all $\lambda \in \mathbb{N}$. Then $0 \longrightarrow \varprojlim M'_\lambda \xrightarrow{\varprojlim \alpha_\lambda} \varprojlim M_\lambda \xrightarrow{\varprojlim \beta_\lambda} \varprojlim M''_\lambda$ is exact.

Proof: Remark (7.82) implies that $\varprojlim \alpha_\lambda$ is injective and that $\varprojlim \beta_\lambda \varprojlim \alpha_\lambda = 0$. Let $x \in \varprojlim M_\lambda$ with $(\varprojlim \beta_\lambda)(x) = 0$. Then for all $\lambda \in \mathbb{N}$ $p_\lambda((\varprojlim \beta_\lambda)(x)) = \beta_\lambda(p_\lambda(x)) = 0$ and there is an $y_\lambda \in M'_\lambda$ with $\alpha_\lambda(y_\lambda) = p_\lambda(x)$. Let $y \in \prod M'_\lambda$ with $p'_\lambda(y) = y_\lambda$. We claim that $y \in \varprojlim M'_\lambda$. For all $\lambda, \mu \in \mathbb{N}$ there is a commutative diagram:

$$\begin{tikzcd} M'_{\lambda+\mu} \arrow[r, "\alpha_{\lambda+\mu}"] \arrow[d, "f'_{\lambda,\lambda+\mu}"'] & M_{\lambda+\mu} \arrow[d, "f_{\lambda,\lambda+\mu}"] \\ M'_\lambda \arrow[r, "\alpha_\lambda"] & M_\lambda \end{tikzcd}$$

Thus $\alpha_\lambda f'_{\lambda,\lambda+\mu}(y_{\lambda+\mu}) = f_{\lambda,\lambda+\mu}\alpha_{\lambda+\mu}(y_{\lambda+\mu}) = f_{\lambda,\lambda+\mu}(p_{\lambda+\mu}(x)) = p_\lambda(x) = \alpha_\lambda(y_\lambda)$. Since α_λ is injective, $y_\lambda = f_{\lambda,\lambda+\mu}(y_{\lambda+\mu})$. Thus $y \in \varprojlim M'_\lambda$ with $(\varprojlim \alpha_\lambda)(y) = x$.

(7.84) Definition: Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of A-modules over $\Lambda = \mathbb{N}$. We say that $\{M_\lambda, f_{\lambda\mu}\}$ satisfies the Mittag-Leffler condition ML if for every λ the decreasing chain of submodules $\{f_{\lambda\mu}(M_\mu) \mid \mu \geq \lambda\}$ stabilizes, i.e. there is an $n_\lambda \in \mathbb{N}$ so that $f_{\lambda n_\lambda}(M_{n_\lambda}) = f_{\lambda\mu}(M_\mu)$ for all $\mu \geq n_\lambda$.

Note: Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system over $\Lambda = \mathbb{N}$ with $f_{\lambda\mu}: M_\mu \longrightarrow M_\lambda$ surjective for all $\lambda, \mu \in \mathbb{N}$. Then $\{M_\lambda, f_{\lambda\mu}\}$ satisfies ML.

(7.85) Proposition: Let $\{M_\lambda\}$ be a direct (an inverse) system of A-modules (rings) over $\mathbb{N}$. and let (n_i) be a strictly increasing sequence of integers. Then

$$\varinjlim M_\lambda \cong \varinjlim M_{n_i} \quad \text{and} \quad \varprojlim M_\lambda = \varprojlim M_{n_i}.$$

Proof: Homework

(7.86) Theorem: Let $\{M_\lambda, f_{\lambda\mu}\}$, $\{M'_\lambda, f'_{\lambda\mu}\}$, $\{M''_\lambda, f''_{\lambda\mu}\}$ be inverse systems of A-modules over $\mathbb{N}$ with $f'_{\lambda\mu}: M'_\mu \longrightarrow M'_\lambda$ surjective for all $\lambda, \mu \in \mathbb{N}$, $\lambda \leq \mu$. Suppose that $\alpha = \{\alpha_\lambda\}: \{M'_\lambda, f'_{\lambda\mu}\} \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ and $\beta = \{\beta_\lambda\}: \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M''_\lambda, f''_{\lambda\mu}\}$ are A-linear maps of inverse systems with $0 \longrightarrow M'_\lambda \xrightarrow{\alpha_\lambda} M_\lambda \xrightarrow{\beta_\lambda} M''_\lambda \longrightarrow 0$ exact for all $\lambda \in \mathbb{N}$. Then

$$0 \longrightarrow \varprojlim M'_\lambda \xrightarrow{\lim \alpha_\lambda} \varprojlim M_\lambda \xrightarrow{\lim \beta_\lambda} \varprojlim M''_\lambda \longrightarrow 0$$

is exact.

Proof: We only have to show that $\varprojlim \beta_\lambda$ is surjective. Let $z = (z_\lambda) \in \varprojlim M''_\lambda$. We construct an $x = (x_\lambda) \in \prod M_\lambda$ inductively so that $x \in \varprojlim M_\lambda$ and $(\varprojlim \beta_\lambda)(x) = z$. Pick $x_0 \in M_0$ with $\beta_0(x_0) = z_0$ and suppose that $x_i \in M_i$ for $0 \leq i \leq n-1$ have been constructed with

(a) $\beta_i(x_i) = z_i$ for $0 \leq i \leq n-1$

(b) $f_{i-1\,i}(x_i) = x_{i-1}$ for $1 \leq i \leq n-1$.

Consider the commutative diagram:

$$\begin{array}{ccccc}
M'_n & \xrightarrow{\alpha_n} & M_n & \xrightarrow{\beta_n} & M''_n \\
{\scriptstyle f'_{n-1,n}} \downarrow & & \downarrow {\scriptstyle f_{n-1,n}} & & \downarrow {\scriptstyle f''_{n-1,n}} \\
M'_{n-1} & \xrightarrow{\alpha_{n-1}} & M_{n-1} & \xrightarrow{\beta_{n-1}} & M''_{n-1}
\end{array}$$

Let $\tilde{x}_n \in M_n$ with $\beta_n(\tilde{x}_n) = z_n$. Then $\beta_{n-1}(f_{n-1,n}(\tilde{x}_n) - x_{n-1}) = \beta_{n-1} f_{n-1,n}(\tilde{x}_n) - \beta_{n-1}(x_{n-1}) = f''_{n-1,n}(z_n) - \beta_{n-1}(x_{n-1}) = z_{n-1} - z_{n-1} = 0$. Hence there is an $y_{n-1} \in M'_{n-1}$ with $\alpha_{n-1}(y_{n-1}) = f_{n-1,n}(\tilde{x}_n) - x_{n-1}$. Since $f'_{n-1,n}$ is surjective there is an $y_n \in M'_n$ so that $f'_{n-1,n}(y_n) = y_{n-1}$. Set $x_n = \tilde{x}_n - \alpha_n(y_n)$. Then:

(a) $\beta_n(x_n) = \beta_n(\tilde{x}_n) - \beta_n\alpha_n(y_n) = \beta_n(\tilde{x}_n) = z_n$

(b) $$\begin{aligned} f_{n-1,n}(x_n) = f_{n-1,n}(\tilde{x}_n - \alpha_n(y_n)) &= f_{n-1,n}(\tilde{x}_n) - f_{n-1,n}\alpha_n(y_n) \\ &= f_{n-1,n}(\tilde{x}_n) - \alpha_{n-1} f'_{n-1,n}(y_n) \\ &= f_{n-1,n}(\tilde{x}_n) - \alpha_{n-1}(y_{n-1}) = x_{n-1}. \end{aligned}$$

Thus with $x = (x_n) \in \prod M_n$, $x \in \varprojlim M_n$ and $(\varprojlim \beta_n)(x) = z$.

(7.87) <u>Remark</u>: Theorem (7.86) also holds true under the (weaker) assumption that $\{M'_\lambda, f'_{\lambda\mu}\}$ satisfies ML.

§7: SOME IDENTITIES

In this section we list some homological identities. The proofs, if not provided, can be found in most books on homological algebra, for example: J. Rotman: An introduction to homological algebra.

(7.88) Remark: Let $\{M_i\}_{i\in I}$ be a family of A-modules and N an A-module. Then

(a) $\mathrm{Hom}_A(\bigoplus M_i, N) \cong \prod \mathrm{Hom}_A(M_i, N)$

(b) $\mathrm{Hom}_A(N, \prod M_i) \cong \prod \mathrm{Hom}_A(N, M_i)$

We know from (6.37) that the tensor product commutes with direct sums:

(c) $(\bigoplus M_i)\otimes_A N \cong \bigoplus (M_i \otimes_A N)$

Since $\bigoplus$ and $\prod$ commute with the formation of homology, the identities of (7.88) extend to the derived functors:

(7.89) Proposition: Let $\{M_i\}_{i\in I}$ be a family of A-modules and N an A-module.

(a) $\mathrm{Ext}^i_A(\bigoplus_{j\in I} M_j, N) \cong \prod_{j\in I} \mathrm{Ext}^i_A(M_j, N)$

(b) $\mathrm{Ext}^i_A(N, \prod_{j\in I} M_j) \cong \prod_{j\in I} \mathrm{Ext}^i_A(N, M_j)$

(c) $\mathrm{Tor}^A_i(\bigoplus_{j\in I} M_j, N) \cong \bigoplus_{j\in I} \mathrm{Tor}^A_i(M_j, N)$.

(7.90) Theorem: Let $A \longrightarrow B$ be a homomorphism of rings so that B is a flat A-module and let M and N be A-modules. Then:

(a) $\mathrm{Tor}^B_i(B\otimes_A M, B\otimes_A N) \cong B\otimes_A \mathrm{Tor}^A_i(M, N)$

(b) If A is Noetherian and M is finitely generated, then

$$\mathrm{Ext}^i_B(B\otimes_A M, B\otimes_A N) \cong B\otimes_A \mathrm{Ext}^i_A(M, N)$$

Proof: We only prove (b). Since M is finitely generated and A is Noetherian, M has a free resolution $F_\bullet$ with F_j finite, say $F_j \cong A^{n_j}$. Thus by (7.89):

$$B\otimes_A \mathrm{Hom}_A(F_j,N) \cong B\otimes_A \mathrm{Hom}_A(A^{n_j},N) \cong B\otimes_A \bigoplus^{n_j} \mathrm{Hom}_A(A,N) \cong B\otimes_A \bigoplus^{n_j} N \cong$$
$$\cong \bigoplus^{n_j} B\otimes_A N \cong \bigoplus^{n_j} \mathrm{Hom}_B(B, B\otimes_A N) \cong \mathrm{Hom}_B(\bigoplus^{n_j} B, B\otimes_A N) \cong$$
$$\cong \mathrm{Hom}_B(B\otimes_A F_j, B\otimes_A N),$$

and this isomorphism is natural. Hence there is an isomorphism of complexes $B\otimes_A \mathrm{Hom}_A(F_\cdot, N) \cong \mathrm{Hom}_B(B\otimes_A F_\cdot, B\otimes_A N)$. Since B is A-flat, $B\otimes_A F_\cdot$ is a free B-resolution of $B\otimes_A M$. Hence

$$\begin{aligned}
\mathrm{Ext}^i_B(B\otimes_A M, B\otimes_A N) &= H^i(\mathrm{Hom}_B(B\otimes_A F_\cdot, B\otimes_A N)) \\
&\cong H^i(B\otimes_A \mathrm{Hom}_A(F_\cdot, N)) \\
&\cong B\otimes_A H^i(\mathrm{Hom}_A(F_\cdot, N)) \qquad (\text{since } B \text{ is } A\text{-flat}) \\
&= B\otimes_A \mathrm{Ext}^i_A(M,N).
\end{aligned}$$

From the universal property of inverse limits one can derive the following properties:

(7.91) <u>Proposition</u>: Let $\{M_i\}$ be a direct (inverse) system of A-modules, N an A-module.

(a) $\mathrm{Hom}_A(\varinjlim M_i, N) \cong \varprojlim \mathrm{Hom}_A(M_i, N)$

(b) $\mathrm{Hom}_A(N, \varprojlim M_i) \cong \varprojlim \mathrm{Hom}_A(N, M_i)$.

Let $I \subseteq A$ be an ideal and $\Gamma_I(-)$ the torsion functor. Notice that $\Gamma_I(-)$ is an additive functor. For every A-module M:

$$\begin{aligned}
\Gamma_I(M) &= \{m\in M \mid I^n m = 0 \text{ for some } n\in \mathbb{N}\} \\
&= \varinjlim (0 :_M I^n) \\
&\cong \varinjlim \mathrm{Hom}_A(A/I^n, M).
\end{aligned}$$

The right derived functors of $\Gamma_I(-)$ are the <u>local cohomology</u> functors. Notation: $H^i_I(-) = R^i\Gamma_I(-)$. $H^i_I(-)$ are additive functors with $H^0_I(-) = \Gamma_I(-)$.

(7.92) <u>Proposition</u>: $H^i_I(M) \cong \varinjlim \mathrm{Ext}^i_A(R/I^n, M)$ and this isomorphism is natural.

<u>Proof</u>: $H^i_I(M) \cong H^i_I(\Gamma_I(I^\cdot_M)) \cong H^i(\varinjlim \mathrm{Hom}_A(R/I^n, I^\cdot_M))$

$$\cong \varinjlim H^i(\mathrm{Hom}_A(A/I^n, I^\bullet_M)) \quad \text{since } \varinjlim \text{ preserves exactness}$$
$$\cong \varinjlim \mathrm{Ext}^i_A(A/I^n, M).$$

(7.93) <u>Theorem</u>: If $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules, then there is a long exact sequence:

$$0 \to H^0_I(M') = \Gamma_I(M') \longrightarrow H^0_I(M) = \Gamma_I(M) \longrightarrow H^0_I(M'') = \Gamma_I(M'') \longrightarrow H^1_I(M') \longrightarrow H^1_I(M) \longrightarrow \cdots$$

Furthermore this exact sequence is natural.

§8: MORE ON FLATNESS

A homomorphism of rings $\varphi: A \to B$ is called flat, faithfully flat, if B as an A-module is flat, faithfully flat, respectively. Equivalently one says that B is a flat, faithfully flat A-algebra.

(7.94) Proposition: Let B be an A-algebra and M an B-module.
(a) If B is flat (faithfully flat) over A and M is flat (faithfully flat) over B, then M is flat (faithfully flat) over A.
(b) Let M be faithfully flat over B. Then B is flat (faithfully flat) over A if and only if M is flat (faithfully flat) over A.

Proof: Let $\mathcal{L}_\cdot: \quad \dots \to N' \to N \to N'' \to \dots$ be a sequence of A-modules and A-linear maps. Then $(\mathcal{L}_\cdot \otimes_A B) \otimes_B M \cong \mathcal{L}_\cdot \otimes_A (B \otimes_B M) \cong \mathcal{L}_\cdot \otimes_A M$.

(7.95) Lemma: Let $\varphi: A \to B$ be a homomorphism of rings, $q \in \mathrm{Spec}(B)$, $p = \varphi^{-1}(q)$, M an B-module and N an A-module. Then $\mathrm{Tor}_i^A(M,N)_q \cong \mathrm{Tor}_i^{A_p}(M_q, N_p)$.

Proof: Let $F_\cdot$ be a free A-resolution of N. Since A_p is A-flat, $A_p \otimes_A F_\cdot$ is a free A_p-resolution of $A_p \otimes_A N \cong N_p$. Then

$$\begin{aligned}
\mathrm{Tor}_i^A(M,N)_q &\cong B_q \otimes_B \mathrm{Tor}_i^A(M,N) \\
&= B_q \otimes_B H_i(M \otimes_A F_\cdot) \\
&\cong H_i(B_q \otimes_B (M \otimes_A F_\cdot)) \qquad \text{since } B_q \text{ is } B\text{-flat} \\
&\cong H_i(M_q \otimes_A F_\cdot) \\
&\cong H_i((M_q \otimes_{A_p} A_p) \otimes_A F) \\
&\cong H_i(M_q \otimes_{A_p} (A_p \otimes_A F_\cdot)) \\
&\cong \mathrm{Tor}_i^{A_p}(M_q, N_p).
\end{aligned}$$

By (6.67) an A-module M is flat if and only if M_m is A_m-flat for all $m \in m\text{-Spec}(A)$.

(7.96) <u>Proposition</u>: Let $\varphi: A \longrightarrow B$ be a homomorphism of rings, M an B-module, and for $Q \in \text{Spec}(B)$ write $P = \varphi^{-1}(Q) \in \text{Spec}(A)$. The following are equivalent:

(a) M is A-flat

(b) M_Q is A_P-flat for all $Q \in \text{Spec}(B)$

(c) M_Q is A_P-flat for all $Q \in m\text{-Spec}(B)$.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (7.47) we have to show that for every A_P-ideal J, $\text{Tor}_1^{A_P}(M_Q, A_P/J) = 0$. Since $J = I_P$ for some A-ideal I, by (7.95) $\text{Tor}_1^{A_P}(M_Q, A_P/J) \cong \text{Tor}_1^{A_P}(M_Q, (A/I)_P) \cong \text{Tor}_1^{A}(M, A/I)_Q$. The latter module vanishes by (7.47).

(c) $\Rightarrow$ (a): By (7.47) we have to show that for every A-ideal I, $\text{Tor}_1^A(M, A/I) = 0$. By (7.95) for every $Q \in m\text{-Spec}(B)$, $\text{Tor}_1^A(M, A/I)_Q \cong \text{Tor}_1^{A_P}(M_Q, A_P/I_P) = 0$ by (7.47). Since the B-module $\text{Tor}_1^A(M, A/I)$ vanishes locally at every maximal ideal of B, by the Local-global principle $\text{Tor}_1^A(M, A/I) = 0$.

Let $\varphi: A \longrightarrow B$ be a homomorphism of rings and let ${}^a\varphi: \text{Spec}(B) \longrightarrow \text{Spec}(A)$ denote the induced map given by ${}^a\varphi(Q) = \varphi^{-1}(Q)$. Note that ${}^a\varphi$ is a continuous map. For $P \in \text{Spec}(A)$, write $k(P) = A_P/PA_P$. Then $k(P) \otimes_A B \cong S^{-1}B/PS^{-1}B$ where $S = A - P$. Then:

$$\begin{aligned} \text{Spec}(k(P) \otimes_A B) &= \{ Q(k(P) \otimes_A B) \mid Q \in \text{Spec}(B),\ Q \cap \varphi(S) = \emptyset,\ Q \supseteq \varphi(P) \} \\ &= \{ Q(k(P) \otimes_A B) \mid Q \in \text{Spec}(B),\ \varphi^{-1}(Q) = P \} \\ &\underset{\text{homeo}}{\cong} \{ Q \mid Q \in \text{Spec}(B),\ \varphi^{-1}(Q) = P \} = ({}^a\varphi)^{-1}(P). \end{aligned}$$

$\text{Spec}(k(P) \otimes_A B)$ is called the <u>fiber</u> <u>over</u> P.

(7.97) <u>Theorem</u>: Let $\varphi: A \longrightarrow B$ be a homomorphism of rings and let M be an B-module.

(a) If M is faithfully flat over A, then ${}^a\varphi(\text{Supp}(M)) = \text{Spec}(A)$.

(b) Let M be a finitely generated B-module. M is faithfully flat over A if and only if M is A-flat and $m\text{-Spec}(A) \subseteq {}^a\varphi(\mathrm{Supp}_B M)$.

Proof. Let $P \in \mathrm{Spec}(A)$ and set $T = k(P) \otimes_A B$. Then $T \otimes_B M = k(P) \otimes_A B \otimes_B M \cong k(P) \otimes_A M$ and $k(P) \otimes_A M \neq 0$, since M is faithfully flat over A. Thus for the T-module $T \otimes_B M$: $\mathrm{Supp}_T(k(P) \otimes_A M) \neq \emptyset$. Pick $Q_0 \in \mathrm{Supp}_T(k(P) \otimes_A M)$ and let Q be the preimage of Q_0 in B. Then $P = \varphi^{-1}(Q)$. Since $(k(P) \otimes_A M)_{Q_0}$ is a homomorphic image of M_Q, we have that $M_Q \neq 0$ and $Q \in \mathrm{Supp}_B(M)$.

(b) We have to show that if M is flat over A and $m\text{-Spec}(A) \subseteq {}^a\varphi(\mathrm{Supp}(M))$ then M is faithfully flat over A. By (6.55) it suffices to show that for every $m \in m\text{-Spec}(A)$ $M \neq mM$. By assumption there is a $Q \in \mathrm{Spec}(B)$ with $\varphi^{-1}(Q) = m$ and $M_Q \neq 0$. Suppose that $M = mM$, then $M = QM$, since $\varphi(m) \subseteq Q$, and therefore $M_Q = QM_Q$. M_Q is a finitely generated module over the local ring B_Q and by Nakayama's Lemma $M_Q = 0$, a contradiction.

(7.98) Corollary: Let $\varphi: A \to B$ be a homomorphism of rings. The following are equivalent:

(a) B is faithfully flat over A.

(b) B is flat over A and φ satisfies lying over (i.e. ${}^a\varphi$ is surjective).

(c) B is flat over A and for all $m \in m\text{-Spec}(A)$ there exists $Q \in \mathrm{Spec}(B)$ lying over m.

Let (A, m) and (B, n) be local rings. A homomorphism of rings $\varphi: A \to B$ is called local if $\varphi(m) \subseteq n$. Then $\varphi^{-1}(n) = m$. By (7.98), a local homomorphism is faithfully flat if and only if it is flat.

A homomorphism of rings $\varphi: A \to B$ is said to satisfy going down if for every chain $P_0 \supset P_1 \supset \dots \supset P_n$ with $P_i \in \mathrm{Spec}(A)$ and every $Q_0 \in \mathrm{Spec}(B)$ with $\varphi^{-1}(Q_0) = P_0$ there exists a chain $Q_0 \supset Q_1 \supset \dots \supset Q_n$ with $Q_i \in \mathrm{Spec}(B)$ and $\varphi^{-1}(Q_i) = P_i$. If φ satisfies going down then for every A-ideal I, $\mathrm{ht}\, IB \geq \mathrm{ht}\, I$.

(7.99) Theorem: Let $\varphi: A \to B$ be a flat homomorphism of rings. Then φ satisfies going down.

<u>Proof</u>: By induction it suffices to consider a chain of length one, $P_0 \supset P_1$, $P_i \in \mathrm{Spec}(A)$. Let $Q_0 \in \mathrm{Spec}(B)$ with $\varphi^{-1}(Q_0) = P_0$. By (7.96) B_{Q_0} is flat over A_{P_0}. Since the homomorphism $A_{P_0} \longrightarrow B_{Q_0}$ is local, it is faithfully flat and satisfies lying over. Thus there is a $Q_1' \in \mathrm{Spec}(B_{Q_0})$ lying over $P_1 A_{P_0}$. Let Q_1 be the preimage of Q_1' in B. Then $Q_1 \in \mathrm{Spec}(B)$, Q_1 lies over P_1, and $Q_1 \subset Q_0$.

(7.100) <u>Theorem</u>:

(a) Let A be a ring, M a flat A-module, N an A-module and N_1, N_2 submodules of N. Then as submodules of $M \otimes N$ one has $(N_1 \cap N_2) \otimes M = (N_1 \otimes M) \cap (N_2 \otimes M)$.

(b) Let $\varphi : A \longrightarrow B$ be a flat homomorphism of rings and I_1, I_2 A-ideals. Then $(I_1 \cap I_2)B = I_1 B \cap I_2 B$.

(c) If in addition I_2 is finitely generated, then $(I_1 : I_2)B = I_1 B : I_2 B$.

<u>Proof</u>: (a) Consider the A-linear map $f : N \longrightarrow N/N_1 \oplus N/N_2$ defined by $f(x) = (x + N_1, x + N_2)$. Then $\ker(f) = N_1 \cap N_2$ and $0 \longrightarrow N_1 \cap N_2 \longrightarrow N \xrightarrow{f} N/N_1 \oplus N/N_2$ is exact. Since M is flat, the sequence $0 \to (N_1 \cap N_2) \otimes M \longrightarrow N \otimes M \xrightarrow{f \otimes M} N\otimes M/N_1 \otimes M \oplus N \otimes M/N_2 \otimes M$ is exact. Thus $(N_1 \cap N_2) \otimes M = (N_1 \otimes M) \cap (N_2 \otimes M)$.

(b) If B is flat over A, for every A-ideal I the natural map $I \otimes_A B \longrightarrow IB$ is an isomorphism. The statement follows from (a).

(c) First consider the case where $I_2 = (a)$ is principal. Define $f : A \xrightarrow{a} A/I_1$ by $f(x) = ax + I_1$. Then $\ker(f) = I_1 : (a)$. Thus $0 \to I_1 : (a) \longrightarrow A \xrightarrow{a} A/I_1$ is exact. Tensoring with $\otimes_A B$ and using the A-flatness of B yields an exact sequence:
$0 \longrightarrow (I_1 : (a))B \longrightarrow B \xrightarrow{a} B/I_1 B$. Thus $(I_1 : (a))B = I_1 B : (a)B$.

If $I_2 = (a_1, \ldots, a_n)$ is a finitely generated A-ideal, then $I_1 : I_2 = \bigcap_{i=1}^{n} (I_1 \cap (a_i))$.
By the above case: $(I_1 : (a_i))B = I_1 B : (a_i)B$ and therefore with (b): $(I_1 : I_2)B = (\bigcap_{i=1}^{n} (I_1 : (a_i)))B = \bigcap_{i=1}^{n} (I_i : (a_i))B = \bigcap_{i=1}^{n} (I_1 B : (a_i)B) = I_1 B : I_2 B$.

(7.101) <u>Theorem</u>: Let $\varphi: A \to B$ be a faithfully flat homomorphism of rings.

(a) For every A-module M, the map $M \cong M \otimes_A A \xrightarrow{M \otimes \varphi} M \otimes_A B$ is injective.

(b) φ is injective.

(c) $\varphi^{-1}(IB) = I$ for every A-ideal I.

<u>Proof</u>: (a) Let $\psi: M \to M \otimes_A B$ be given by $\psi(m) = m \otimes 1$ and let $U = \ker\psi$. Since B is flat over A, $U \otimes_A B \subseteq M \otimes_A B$ and $U \otimes_A B = 0$ by definition of U. Since B is faithfully flat over A, $U = 0$ and ψ is injective.

(b) Apply (a) with $M = A$.

(b) Apply (a) with $M = A/I$. Then $\psi: A/I \to S/IS$ is injective with $0 = \ker\psi = \varphi^{-1}(IS)/I$.

(7.102) <u>Theorem</u>: Let M be an A-module. The following are equivalent:

(a) M is flat

(b) $\mathrm{Tor}_1^A(M, A/I) = 0$ for every finitely generated A-ideal I.

(c) $I \otimes_A M \xrightarrow{\simeq} IM$ via the natural map for every finitely generated A-ideal I.

<u>Proof</u>: (a) $\Rightarrow$ (b): (7.46)

(b) $\Rightarrow$ (c): Follows from the exact sequence $0 \to I \to A \to A/I \to 0$

(c) $\Rightarrow$ (a): Let $I \subseteq A$ be any ideal. By Homework, it suffices to show that $I \otimes_A M \xrightarrow{\simeq} IM$ via the natural map. Since $I = \bigcup_{\lambda \in \Lambda} I_\lambda$ with $\{I_\lambda \mid \lambda \in \Lambda\}$ the set of all finitely generated A-ideals contained in I. $\{I_\lambda \mid \text{incl.}\}$ is a direct system with $I \cong \varinjlim I_\lambda$. Also, $\{A_\lambda = A\,;\ \mathrm{id}\}$ is a direct system with $A \cong \varinjlim A_\lambda$. By assumption (c), the natural maps $I_\lambda \otimes_A M \to A_\lambda \otimes_A M \cong M$ are injective. Thus the induced map $\varinjlim (I_\lambda \otimes_A M) \to \varinjlim (A_\lambda \otimes_A M)$ is injective by (7.76). By (7.77):

$\varinjlim (I_\lambda \otimes_A M) \cong (\varinjlim I_\lambda) \otimes_A M \cong I \otimes_A M$ and $\varinjlim (A_\lambda \otimes_A M) \cong (\varinjlim A_\lambda) \otimes_A M \cong A \otimes_A M \cong M$.

Thus the natural map $I \otimes_A M \to M$ is injective and $I \otimes_A M \xrightarrow{\simeq} IM$ via the natural map.

(7.103) <u>Theorem</u>: (Equitorial criterion for flatness) Let M be an A-module. Consider the following condition $(*)$: Given an $r \times n$ system of linear equations:

$$\sum_{j=1}^{n} a_{ij}\, x_j = 0 \ , \quad a_{ij} \in A \quad \text{and} \quad x_j \in M$$

there exists an integer s, elements $y_k \in M$ with $1 \leq k \leq s$, and elements $b_{jk} \in A$ with $1 \leq j \leq n$, $1 \leq k \leq s$ so that

$$\sum_{j=1}^{n} a_{ij}\, b_{jk} = 0 \quad \text{for every } k \quad \text{and} \quad x_j = \sum_{k=1}^{s} b_{jk}\, y_k$$

i.e. every solution in M is a linear combination of solutions in A with coefficients in M. Then:

(a) If M is flat then $(*)$ holds.

(b) If $(*)$ holds for $r=1$, then M is flat.

<u>Proof</u>: Consider $\varphi : A^n \longrightarrow A^r$ given by the $r \times n$ matrix (a_{ji}) and let $K = \ker(\varphi) \subseteq A^n$. The vector $(b_1, \ldots, b_n) \in A^n$ is in K if and only if $\sum a_{ij} b_j = 0$. Consider $\varphi \otimes M : A^n \otimes M = M^n \longrightarrow A^r \otimes M = M^r$. Then $(x_1, \ldots, x_n) \in M^n$ is in $\ker(\varphi \otimes M)$ if and only if $\sum a_{ij} x_j = 0$. $(*)$ means that $\ker(\varphi \otimes M)$ is contained in the image of $K \otimes M$ in $A^n \otimes M \cong M^n$. Thus $(*)$ is equivalent to $(**)$: If $0 \longrightarrow K \longrightarrow A^n \longrightarrow A^r$ is exact, then $K \otimes M \longrightarrow A^n \otimes M \longrightarrow A^r \otimes M$ is exact.

(a) If M is flat $(**)$ holds.

(b) By (7.102) we have to show that $I \otimes M \xrightarrow{\sim} IM$ via the natural map for every finitely generated A-ideal I. Let I be generated by n elements. Then there is an exact sequence $0 \longrightarrow K \longrightarrow A^n \longrightarrow A \longrightarrow A/I \longrightarrow 0$. By $(**)$ for $r=1$ the sequence $K \otimes M \longrightarrow M^n \longrightarrow M$ is exact. Thus $K \otimes M \xrightarrow{f} M^n \longrightarrow IM \longrightarrow 0$ is exact. From $K \longrightarrow A^n \longrightarrow I \longrightarrow 0$ and the right exactness of $\otimes_A M$ we obtain an exact sequence $K \otimes M \xrightarrow{f} M^n \longrightarrow I \otimes M \longrightarrow 0$. Thus $I \otimes M \cong \operatorname{coker} f \cong IM$.