# CHAPTER XII: REGULARITY CRITERIA

## §1: AN EXTENSION OF THEOREM (5.4)

Recall the following theorem from Chapter V:

Theorem: (5.4) Let $k$ be a field, $S = k[x_1, \ldots, x_n]$ the polynomial ring over $k$, and $I, P \subseteq S$ ideals with $P$ a prime ideal and $I \subseteq P$. Set $R = S_P$, $PR = M$, $A = R/IR$, $MA = m$, $R/M = A/m = K$. Suppose that $\operatorname{ht} I = r$ and $IR = (f_1, \ldots, f_t)$ with $f_1, \ldots, f_t \in S$. Then the following conditions are equivalent:

(a) $\operatorname{rank}(\partial f_i/\partial x_j \bmod P) = r$

(b) $A$ is smooth over $k$

(c) The module of differentials $\Omega_{A/k}$ is a free $A$-module of rank $n-r$.

(d) $A$ is a domain, its field of fractions $Q(A)$ is separable over $k$, and the module of differentials $\Omega_{A/k}$ is a free $A$-module.

(12.1) Theorem: Assumptions as in Theorem (5.4). Then conditions (a)–(d) are equivalent to:

(e) $A$ is $m$-smooth over $k$.

Proof: Obviously, (b) $\Rightarrow$ (e). We want to show (e) $\Rightarrow$ (a). By (8.33) $A$ is geometrically regular over $k$, in particular, $A$ is a regular local ring. Then $IR$ is generated by part of a regular system of parameters of $R$, say $IR = (f_1, \ldots, f_r)$ and $f_1 + M^2, \ldots, f_r + M^2 \in M/M^2$ are linearly independent over $K = R/M$.

Let $P \subseteq k$ be a prime field. Then $K$ is separable (= smooth) over $P$ and by (2.15) the sequence

$$0 \longrightarrow M/M^2 \longrightarrow \Omega_R \otimes_R K \longrightarrow \Omega_K \longrightarrow 0$$

is exact. Note that

$$\Omega_S = (\Omega_k \otimes_k S) \oplus \bigoplus_{i=1}^{n} S\,dx_i$$

and set $F = \bigoplus_{i=1}^{n} S\,dx_i$. Then

$$\Omega_R = (\Omega_k \otimes_k R) \oplus (F \otimes_S R) \quad \text{and} \quad \Omega_R \underset{R}{\otimes} K = (\Omega_k \otimes_k K) \oplus (F \otimes_S K).$$

By (1.12) there is an exact sequence

$$IR/I^2R \longrightarrow \Omega_R \otimes_R A \longrightarrow \Omega_A \longrightarrow 0.$$

Tensoring with K over A yields an exact sequence

$$(I/I^2) \otimes_A K \xrightarrow{\delta} \Omega_R \otimes_R K \longrightarrow \Omega_A \otimes_A K \longrightarrow 0$$

where $\delta$ is induced by the universal derivation $d: R \longrightarrow \Omega_R$. Since R is the localization of a polynomial ring, R is smooth over k and k is smooth over its prime field P. Thus R is smooth over P and by (9.13) $\delta$ is injective. We have an exact sequence:

$$0 \longrightarrow I/I^2 \otimes_A K \xrightarrow{\delta} (\Omega_k \otimes_k K) \oplus (F \otimes_S K) \xrightarrow{\mu} \Omega_A \otimes_A K \longrightarrow 0.$$

Note that $\mu|_{\Omega_k \otimes K}: \Omega_k \otimes_k K \longrightarrow \Omega_A \otimes_A K$ is the natural map. Since A is m-smooth by (8.30) the natural map $\Omega_k \otimes_k K \longrightarrow \Omega_R \otimes_R K$ is injective. Thus the composition of maps:

$$\sigma = \pi\delta: I/I^2 \otimes_A K \xrightarrow{\delta} (\Omega_k \otimes_k K) \oplus (F \otimes_S K) \xrightarrow{\pi} F \otimes_S K$$

is injective where $\pi$ is the projection. Note that $\sigma(f_i) = \sum_{j=1}^{n} \partial f_i/\partial x_j\, dx_j$ and the subspace $\operatorname{im}(\sigma) = \langle \sigma(f_1), \text{—}, \sigma(f_r) \rangle$ of $F \otimes_S K$ has dimension r. This implies that $\operatorname{rank}(\partial f_i/\partial x_j \bmod P) = r$.

(12.2) <u>Corollary</u>: Let $(R, m, \ell)$ be a local ring which is essentially of finite type over a field k. Then the following conditions are equivalent:

(a) R is smooth over k.

(b) R is m-smooth over k.

(12.3) <u>Remark</u>: Assumptions as in Theorem (12.1). Suppose in addition that $k$ is a field of characteristic $0$. Then by (8.33)

$$\begin{aligned} A \text{ is } m\text{-smooth over } k &\iff A \text{ is geometrically regular over } k \\ &\iff A \text{ is a regular local ring (see (8.32))}. \end{aligned}$$

Hence in the characteristic $0$ case Theorem (12.1) is a regularity criterion for affine rings. The characteristic $p>0$ case is more complicated since regularity does not imply smoothness. For example, if $k \subsetneq L$ is an algebraic insparable field extension, then $L$ is a regular local ring, but $L$ is not smooth over $k$. In the next section we will modify Theorem (12.1) to obtain a regularity criterion for affine rings containing a field of characteristic $p>0$.

## §2: A REGULARITY CRITERION FOR AFFINE RINGS

(12.4) Definition: Let $R$ be a ring, $P \subseteq R$ a prime ideal, and $D_1, \dots, D_s \in \mathrm{Der}(R)$ derivations from $R$ to $R$. If $f_1, \dots, f_t \in R$ let

$$\mathcal{J}(f_1, \dots, f_t; D_1, \dots, D_s)(P) = (D_i f_j \bmod P)_{j=1,\dots,t;\, i=1,\dots,s}$$

denote the $t \times s$ Jacobian matrix with entries in $R/P$.

(12.5) Remark: Let $C$ be a $t \times s$-matrix with entries in a ring $R$ and $P \subseteq R$ a prime ideal of $R$. The rank of $C$ modulo $P$ is the rank of $C$ in the quotient field $Q(R/P)$. Note that rank $C = r$ modulo $P$ if and only if there is an $r \times r$-minor $\tilde{C}$ of $C$ with $\det \tilde{C} \not\equiv 0 \bmod P$ and the determinants of all $(r+1) \times (r+1)$ minors of $C$ are $0$ modulo $P$.

(12.6) Proposition: Let $R$ be a regular local ring, $P \subseteq R$ a prime ideal, and $I \subseteq R$ an ideal with $I \subseteq P$. Suppose that $\mathrm{ht}\, IR_P = r$. Then:

(a) For all $s, t \in \mathbb{N}$ and all $D_1, \dots, D_s \in \mathrm{Der}(R)$; $f_1, \dots, f_t \in I$

$$\mathrm{rank}\, \mathcal{J}(f_1, \dots, f_t; D_1, \dots, D_s)(P) \le r.$$

(b) If there are $D_1, \dots, D_r \in \mathrm{Der}(R)$ and $f_1, \dots, f_r \in I$ so that $\det(D_i f_j) \notin P$ then $IR_P = (f_1, \dots, f_r)R_P$ and the ring $(R/I)_P$ is regular.

Proof: (a) Let $Q \subseteq R$ be a prime ideal with $I \subseteq Q \subseteq P$ and $\mathrm{ht}\, Q = r$. Then

$$\mathrm{rank}\, \mathcal{J}(f_1, \dots, f_t; D_1, \dots, D_s)(P) \le \mathrm{rank}\, \mathcal{J}(f_1, \dots, f_t; D_1, \dots, D_s)(Q).$$

Set $m = QR_Q$ and note that $(R_Q, m)$ is a regular local ring of dimension $r$. Suppose $m = (g_1, \dots, g_r)R_Q = QR_Q$ with $g_i \in Q$. Then for all $1 \le j \le t$:

$$f_j = \sum_{\ell=1}^{r} a_{\ell j}\, g_\ell \quad \text{where } a_{\ell j} \in R_Q \text{ and thus}$$

$$D_i f_j \equiv \sum_{\ell=1}^{r} a_{\ell j} (D_i g_\ell) \bmod QR_Q \quad \text{for all } 1 \le i \le s.$$

This shows that

$$\mathrm{rank}\, \mathcal{J}(f_1, \dots, f_t; D_1, \dots, D_s)(Q) \le \mathrm{rank}\, \mathcal{J}(g_1, \dots, g_r; D_1, \dots, D_s)(Q) \le r.$$

(b) Let $M = PR_P$ and $K = R_P/M = (R/P)_P = k(P)$. Consider the $K$-linear map: $D : M/M^2 \longrightarrow K^r$ defined by $D(h+M^2) = (D_1 h + M, \ldots, D_r h + M)$. If $\det(D_i f_j) \notin M$, then $D(f_1), \ldots, D(f_r)$ are linearly independent in $K^r$. Thus $f_1 + M^2, \ldots, f_r + M^2$ are linearly independent in $M/M^2$ and $f_1, \ldots, f_r$ are part of a regular system of parameters in $R_P$. Therefore $Q = (f_1, \ldots, f_r)R_P$ is a prime ideal of height $r$ in $R_P$ with $R_P/Q$ a regular local ring. Since $Q \subseteq IR_P$ and $\operatorname{ht} IR_P = r$, it follows that $Q = IR_P$.

Let $k$ be a field of characteristic $p>0$, $F \subseteq k$ the prime field contained in $k$, and $\{u_\gamma\}_{\gamma\in\Gamma}$ a $p$-basis of $k$ over $F$. For all $\gamma \in \Gamma$ let $\tilde{d}_\gamma$ denote the map $\tilde{d}_\gamma : \{u_\gamma\}_{\gamma\in\Gamma} \longrightarrow k$ defined by $\tilde{d}_\gamma(u_{\gamma'}) = \delta_{\gamma\gamma'}$. $\tilde{d}_\gamma$ extends uniquely to a derivation $d_\gamma : k \longrightarrow k$.

Let $x_1, \ldots, x_n$ be variables over $k$ and $S = k[x_1, \ldots, x_n]$ the polynomial ring over $k$. Then each $d_\gamma$ extends to a derivation $D_\gamma : S \longrightarrow S$ by $D_\gamma(\sum_{(i)} a_{(i)} x_1^{i_1} \cdots x_n^{i_n}) = \sum_{(i)} d_\gamma(a_{(i)}) x_1^{i_1} \cdots x_n^{i_n}$ where $a_{(i)} \in k$.

(12.7) Theorem: (Zariski) Assumptions as above. Let $P \subseteq S$ be a prime ideal and $I \subseteq S$ an ideal with $I \subseteq P$. Set $R = S_P$ and $A = R/IR$ and suppose that $\operatorname{ht} IR = r$ and that $f_1, \ldots, f_t \in S$ with $I = (f_1, \ldots, f_t)R$. The following conditions are equivalent:

(a) $A$ is a regular local ring.

(b) There are elements $\gamma_1, \ldots, \gamma_m \in \Gamma$ so that
$$\operatorname{rank} J(f_1, \ldots, f_t; D_{\gamma_1}, \ldots, D_{\gamma_m}, \partial/\partial x_1, \ldots, \partial/\partial x_n)(P) = r.$$

(c) There is a subfield $\ell \subseteq k$ with $k^p \subseteq \ell$ and $[k:\ell] < \infty$ so that $\Omega_{A/\ell}$ is a free $A$-module with $\operatorname{rank} \Omega_{A/\ell} = n - r + \operatorname{rank} \Omega_{k/\ell}$.

Proof: First note that $\Omega_S = (\Omega_k \otimes_k S) \oplus \bigoplus_{i=1}^n S\,dx_i$ where $d : S \longrightarrow \Omega_S$ is the universal derivation. Let $d_k : k \longrightarrow \Omega_k$ denote the universal derivation of $k$. Then $\Omega_k$ has basis $\{d_k u_\gamma\}_{\gamma\in\Gamma}$. Hence $\Omega_S$ is a free $S$-module with basis $\{du_\gamma\}_{\gamma\in\Gamma} \cup \{dx_1, \dots, dx_n\}$. For all $\gamma\in\Gamma$ let $f_\gamma : \Omega_S \longrightarrow S$ denote the $S$-linear map defined by $f_\gamma(du_{\gamma'}) = \delta_{\gamma\gamma'}$ and $f_\gamma(dx_i) = 0$ for all $1 \le i \le n$. Obviously, $D_\gamma = f_\gamma d : S \xrightarrow{d} \Omega_S \xrightarrow{f_\gamma} S$ for all $\gamma\in\Gamma$!

(b) $\Rightarrow$ (a): Apply (12.6) $r\times r$-minor of rank $r$ of

$$J(f_1, \dots, f_t; D_{\gamma_1}, \dots, D_{\gamma_m}, \partial/\partial x_1, \dots, \partial/\partial x_n)(P).$$

(a) $\Rightarrow$ (b): If $A$ is a regular local ring, then the ideal $IR$ is generated by part of a regular system of parameters of $R$, say $IR = (f_1, \dots, f_r)R$. (Note that we can choose $f_1, \dots, f_r \in S$.) Moreover, $IR/I^2R$ is a free $R/IR = A$-module of rank $r$.

Put $M = PR$, $m = M/IR \subseteq A$, and $K = R/M = A/m$ and let $F$ denote the prime field contained in $k$. Since $F$ is perfect and $A$ regular, $A$ is geometrically regular over $F$. Thus $A$ is $m$-smooth over $F$. Moreover, since $S$ is smooth over $k$, $S$ is smooth over $F$ and so is its localization $R$. Thus by (9.13) the sequence:

$$(*)\qquad 0 \longrightarrow IR/I^2R \otimes_R K \xrightarrow{d} \Omega_R \otimes_R K \longrightarrow \Omega_A \otimes_R K \longrightarrow 0$$

is exact where $d$ is induced by the universal derivation $d_R : R \longrightarrow \Omega_R$. We use again that

$$\Omega_R \otimes_R K = \Omega_S \otimes_S K \cong (\Omega_k \otimes_k K) \oplus \bigoplus_{i=1}^{n} K\, dx_i$$

and write for all $1 \le i \le r$:

$$f_i = \sum_{(i)} a_{i(i)}\, x_1^{i_1} \cdots x_n^{i_n}$$

where $a_{i(i)} \in k$. Every $a_{i(i)}$ can be written as

$$a_{i(i)} = \sum_{(\gamma)} b_{i(i)(\gamma)}\, u_{\gamma_1}^{j_1} \cdots u_{\gamma_m}^{j_m}$$

where $b_{i(i)(\gamma)} \in k^p$, $\gamma_i \in \Gamma$, and $0 \le j_\ell < p$ for $1 \le \ell \le m$. (Note that we do not assume that $\Gamma$ is countable!) This implies

$$(**)\qquad d_R f_i = \sum_{j=1}^{m} a_{i\gamma_j}\, du_{\gamma_j} + \sum_{s=1}^{n} \partial f_i/\partial x_s\, dx_s$$

where $a_{i\gamma_j} = D_{\gamma_j} f_i \in R$. Since $(*)$ is exact, $df_1, \dots, df_r$ are linearly independent in $\Omega_R \otimes_R K$ implying that the $r$ rows

$$(a_{i\gamma_1}, \ldots, a_{i\gamma_m}, \partial f_i/\partial x_1, \ldots, \partial f_i/\partial x_n) \qquad 1 \le i \le r$$

are linearly independent in $K^{m+n}$. Hence the $r$ rows

$$(D_{\gamma_1} f_i, \ldots, D_{\gamma_m} f_i, \partial f_i/\partial x_1, \ldots, \partial f_i/\partial x_n)$$

are linearly independent modulo $M$ and (b) follows.

(a) $\Rightarrow$ (c): Assume

$$(**) \qquad df_i = \sum_{j=1}^{m} a_{i\gamma_j}\, du_{\gamma_j} + \sum_{s=1}^{n} \partial f_i/\partial x_s\, dx_s$$

with $a_{i\gamma_j} = D_{\gamma_j} f_i \in R$ as before and set

$$\ell = k^p(u_\sigma)_{\sigma \in \Pi - \{\gamma_1, \ldots, \gamma_m\}}.$$

Then $[k:\ell] < \infty$ and $\{u_{\gamma_1}, \ldots, u_{\gamma_m}\}$ is a p-basis of $k$ over $\ell$. In particular, $\Omega_{k/\ell}$ is a $k$-vector space of dimension $m$ with basis $du_{\gamma_1}, \ldots, du_{\gamma_m}$. Since $\Omega_{R/\ell} \otimes_R K = (\Omega_{k/\ell} \otimes_k K) \oplus \bigoplus_{s=1}^{n} K\, dx_s$ the proof of (a) $\Rightarrow$ (b) yields that the sequence

$$0 \longrightarrow (IR/I^2R) \otimes_R K \longrightarrow \Omega_{R/\ell} \otimes_R K \longrightarrow \Omega_{A/\ell} \otimes_A K \longrightarrow 0$$

is exact. Moreover, $\Omega_{R/\ell} \otimes_R K$ is a $K$-vector space of dimension $n+m$.

In the exact sequence

$$IR/I^2R \xrightarrow{d^*} \Omega_{R/\ell} \otimes_R A \longrightarrow \Omega_{A/\ell} \longrightarrow 0$$

$\Omega_{R/\ell} \otimes_R A$ is a finite free $A$-module of rank $n+m$ with basis $du_{\gamma_1}, \ldots, du_{\gamma_m}, dx_1, \ldots, dx_n$. By (a) $\Rightarrow$ (b) $d^* f_1, \ldots, d^* f_r \in \Omega_{R/\ell} \otimes_R A$ are linearly independent modulo $m$. Thus by Nakayama $d^* f_1, \ldots, d^* f_r$ are part of a basis of $\Omega_{R/\ell} \otimes_R A$. This implies that the sequence

$$0 \longrightarrow IR/I^2R \xrightarrow{d^*} \Omega_{R/\ell} \otimes_R A \longrightarrow \Omega_{A/\ell} \longrightarrow 0$$

is split exact. $\Omega_{A/\ell}$ is a free $A$-module with

$$\operatorname{rank} \Omega_{A/\ell} = \operatorname{rank}(\Omega_{R/\ell} \otimes_R A) - r = n+m-r = n-r + \operatorname{rank} \Omega_{k/\ell}.$$

(c) $\Rightarrow$ (a): Consider the exact sequence

$$IR/I^2R \xrightarrow{d^*} \Omega_{R/\ell} \otimes_R A \longrightarrow \Omega_{A/\ell} \longrightarrow 0$$

where $\Omega_{R/\ell} \otimes_R A$ and $\Omega_{A/\ell}$ are free $A$-modules with

$$\operatorname{rank} \Omega_{R/\ell} \otimes_R A - \operatorname{rank} \Omega_{A/\ell} = r.$$

Thus $\operatorname{im}(d^*)$ is a direct summand of $\Omega_{R/\ell} \otimes_R A$, more precisely, $\operatorname{im}(d^*)$ is a free $A$-module of rank $r$. Let $f_1, \ldots, f_r \in IR$ (and $f_1, \ldots, f_r \in S$) with $\operatorname{im}(d^*) = \bigoplus_{i=1}^{r} A d^* f_i$. Since $d^* f_1, \ldots, d^* f_r$ can be extended to a basis of $\Omega_{R/e} \otimes_R A$, by Nakayama the images $df_1, \ldots, df_r \in \Omega_{R/e}$ can be extended to a basis of $\Omega_{R/e}$.

Since $\Omega_{R/e}$ is a free $R$-module of rank $n+m$ with $m = \operatorname{rank} \Omega_{k/e}$, $\operatorname{Der}_e(R) \cong \operatorname{Hom}_R(\Omega_{R/e}, R) = \Omega_{R/e}$ is a free $R$-module of rank $n+m$. Let $\omega_1, \ldots, \omega_{n+m}$ be a basis of $\Omega_{R/e}$ and let $h_i : \Omega_{R/e} \longrightarrow R$ be the $R$-linear map defined by $h_i(\omega_j) = \delta_{ij}$. Each $h_i$ corresponds to the $e$-derivation $D_i = h_i d : R \xrightarrow{d} \Omega_{R/e} \xrightarrow{h_i} R$. Then $D_1, \ldots, D_{n+m}$ is a basis of $\operatorname{Der}_e(R)$. Consider the $R$-linear isomorphism

$$\sigma : \Omega_{R/e} \xrightarrow{\cong} R^{n+m}$$

defined by $\sigma(\omega) = (h_1(\omega), \ldots, h_{n+m}(\omega))$. Since $df_1, \ldots, df_r$ are part of a basis of $\Omega_{R/e}$, $\sigma(df_1), \ldots, \sigma(df_r)$ are part of a basis of $R^{n+m}$ where

$$\sigma(df_i) = (h_1 d f_i, \ldots, h_{n+m} d f_i) = (D_1 f_i, \ldots, D_{n+m} f_i).$$

Hence the rows

$$(D_1 f_1 + M, \ldots, D_{n+m} f_1 + M)$$
$$\vdots$$
$$(D_1 f_r + M, \ldots, D_{n+m} f_r + M)$$

are linearly independent in $K^{n+m}$ and there is an $r \times r$-minor of the matrix $(D_i f_j + M)_{i=1,\ldots,n+m;\, j=1,\ldots,r}$, say $(D_i f_j + M)_{i,j=1,\ldots,r}$, so that

$$\det (D_i f_j + M)_{i,j=1,\ldots,r} \neq 0.$$

Therefore $\det (D_i f_j)_{i,j=1,\ldots,r} \notin M$ and by (12.6) the ring $R/IR = A$ is regular with $IR = (f_1, \ldots, f_r)R$.

Let $k$ be a field, $S = k[x_1, \ldots, x_n]$ the polynomial ring and $I \subseteq S$ an ideal. Set $B = S/I$ and consider the following subsets of $\operatorname{Spec}(B)$:
$U = \{P \in \operatorname{Spec}(B) \mid B_P \text{ is } 0\text{-smooth over } k\}$ and $\operatorname{Reg}(B) = \{P \in \operatorname{Spec}(B) \mid B_P \text{ is a RLR}\}$.

(12.8) Corollary: Assumptions as above. Then $\mathcal{U}$ and $\mathrm{Reg}(B)$ are open in $\mathrm{Spec}(B)$.

Proof: For every prime ideal $P \subseteq S$ with $I \subseteq P$ we want to apply the Jacobian criteria of (5.4) and (12.7) to the ring $B_P$. For different prime ideals $P, Q \subseteq S$ with $I \subseteq P$ and $I \subseteq Q$ it may happen that $\mathrm{ht}(IS_P) \neq \mathrm{ht}(IS_Q)$. We first reduce to open subsets of $\mathrm{Spec}(B)$ which contain exactly one minimal prime of $B$.

Let $\mathrm{Spec}(B) = V_1 \cup \dots \cup V_h$ with $V_i$ the irreducible components of $\mathrm{Spec}(B)$. For all $1 \le i \le h$, $V_i = V(q_i)$ where $q_i \subseteq B$ is a minimal prime of $B$. For all $P \in V_i \cap V_j$ with $i \neq j$, the ring $B_P$ has at least two different minimal prime ideals and $P \notin \mathrm{Reg}(B)$. (Notice that $\mathcal{U} \subseteq \mathrm{Reg}(B)$.) Let $W = \bigcup_{i \neq j} (V_i \cap V_j)$, then $\mathcal{U} \subseteq \mathrm{Reg}(B) \subseteq \mathrm{Spec}(B) - W$ and $\mathrm{Spec}(B) - W$ is open in $\mathrm{Spec}(B)$. Moreover,

$$\mathrm{Spec}(B) - W = \dot{\bigcup}_{i=1}^{h} V_i \cap (\mathrm{Spec}(B) - W),$$

that is, $\mathrm{Spec}(B) - W$ is the disjoint union of the open subsets $V_i \cap (\mathrm{Spec}(B) - W)$. Hence it suffices to show: For all $1 \le i \le h$ the sets $[V_i \cap (\mathrm{Spec}(B) - W)] - \mathcal{U}$ and $[V_i \cap (\mathrm{Spec}(B) - W)] - \mathrm{Reg}(B)$ are closed in $V_i \cap (\mathrm{Spec}(B) - W)$.

Fix an $i \in \{1, \dots, h\}$ and assume that $\dim V_i = n - r = \dim(B/q_i)$ where $q_i \subseteq B$ is a minimal prime ideal. If $P \in V_i \cap (\mathrm{Spec}(B) - W)$ is a prime ideal and $P' \subseteq S$ its contraction to $S$ then $\mathrm{ht}\, IS_P = r$. Suppose that $I = (f_1, \dots, f_t) \subseteq S$.

For $\mathcal{U}$: Let $M_1, \dots, M_\lambda$ denote the $r \times r$-minors of the Jacobian matrix $(\partial f_i / \partial x_j)_{i=1,\dots,t;\, j=1,\dots,n}$. Then by (5.4):

$$[V_i \cap (\mathrm{Spec}(B) - W)] - \mathcal{U} = V(\det M_1 + I, \dots, \det M_\lambda + I) \cap (\mathrm{Spec}(B) - W) \cap V_i$$

and $[V_i \cap (\mathrm{Spec}(B) - W)] - \mathcal{U}$ is closed in $V_i \cap (\mathrm{Spec}(B) - W)$.

For $\mathrm{Reg}(B)$: Let $\{u_\gamma\}_{\gamma \in \Gamma}$ be a $p$-basis of $k$ over the prime field $F$ and let for all $\gamma \in \Gamma$ $D_\gamma : S \longrightarrow S$ denote the derivations extended from the maps $\tilde{d}_\gamma : \{u_\gamma\}_{\gamma\in\Gamma} \longrightarrow k$ defined by $\tilde{d}_\gamma(u_{\gamma'}) = \delta_{\gamma\gamma'}$ as in the section before Theorem (12.7). For all finite subsets $\{\gamma_1, \ldots, \gamma_m\} \subseteq \Gamma$ (and all $m \in \mathbb{N}$!) consider the Jacobian matrices

$$(*) \qquad J(f_1, \ldots, f_t ; D_{\gamma_1}, \ldots, D_{\gamma_m}, \partial/\partial x_1, \ldots, \partial/\partial x_n).$$

Let $J \subseteq B$ be the ideal generated by the determinants of all $r\times r$-minors of all matrices of type $(*)$ (modulo $I$). Then

$$[V_i \cap (\mathrm{Spec}(B) - W)] \cap \mathrm{Reg}(B) = V(J) \cap [V_i \cap (\mathrm{Spec}(B) - W)]$$

and $[V_i \cap (\mathrm{Spec}(B) - W)] \cap \mathrm{Reg}(B)$ is closed in $[V_i \cap (\mathrm{Spec}(B) - W)]$.

## §3: WEAK JACOBIAN CRITERIA

Recall: Let $R$ be an integral domain, $K = Q(R)$ its field of fractions, and $M$ an $R$-module. The rank of $M$ over $R$, $\operatorname{rank}_R M$, is the dimension of the $K$-vector space $M \otimes_R K$.

(12.9) Lemma: Let $R$ be an integral domain and $k \subseteq R$ a subring of $R$. If the $k$-derivations $D_1, \ldots, D_n \in \operatorname{Der}_k(R)$ are linearly independent over $R$, then there are elements $a_1, \ldots, a_n \in R$ with $\det(D_i(a_j))_{i,j=1,\ldots,n} \neq 0$.

Proof: Let $K = Q(R)$ be the quotient field of $R$. Every $k$-derivation $D \in \operatorname{Der}_k(R)$ extends uniquely to a $k$-derivation of $\operatorname{Der}_k(K)$ also denoted by $D$. Thus there is an embedding of $R$-modules $\operatorname{Der}_k(R) \subseteq \operatorname{Der}_k(K)$ and the derivations $D_1, \ldots, D_n$ are linearly independent in the $K$-vector space $\operatorname{Der}_k(K)$. Consider the map $d: K \longrightarrow K^n$ defined by $d(a) = (D_1(a), \ldots, D_n(a))$ and let $V = (\operatorname{im} d)K \subseteq K^n$. First we want to show that $V = K^n$. Let $p_i: K^n \longrightarrow K$ denote the projection on the $i$-th component. Then $p_i d = D_i$ for all $1 \leq i \leq n$. If $\dim V \leq n-1$, then the $K$-linear maps $p_1|_V, \ldots, p_n|_V$ are linearly dependent in $\operatorname{Hom}_K(V, K) \cong V$ and there are elements $b_i \in R$, not all $b_i = 0$, with $\sum_{i=1}^n b_i p_i|_V = 0$. Then $(\sum_{i=1}^n b_i p_i|_V) d = \sum_{i=1}^n b_i D_i = 0$, a contradiction since $D_1, \ldots, D_n$ linearly independent in $\operatorname{Der}_k(R)$.

Hence $V = K^n$ and there are elements $c_1, \ldots, c_n \in K$ so that the rows $(D_1(c_1), \ldots, D_n(c_1)), \ldots, (D_1(c_n), \ldots, D_n(c_n))$ form a basis of $K^n$. Let $t, b_1, \ldots, b_n \in R$ with $t \neq 0$ and $c_i = b_i/t$ for all $1 \leq i \leq n$. Then $D_i(c_j) = D_i(b_j/t) = 1/t^2 (t D_i(b_j) - b_j D_i(t))$ and the matrix

$$\begin{pmatrix} t D_1(b_1) - b_1 D_1(t) & \cdots\cdots & t D_n(b_1) - b_1 D_n(t) \\ & & \\ t D_1(b_n) - b_n D_1(t) & \cdots\cdots & t D_n(b_n) - b_n D_n(t) \end{pmatrix}$$

has rank $n$. Since $t \neq 0$, the rank of the $(n+1) \times n$ matrix

$$\begin{bmatrix} D_1(b_1) & \cdots\cdots\cdots & D_n(b_1) \\ & & \\ D_1(b_n) & \cdots\cdots\cdots & D_n(b_n) \\ D_1(t) & \cdots\cdots\cdots & D_n(t) \end{bmatrix}$$

is also $n$ and there are elements $a_1, \text{—}, a_n \in R$ with $\det(D_i(a_j)) \neq 0$.

(12.10) <u>Lemma</u>: Let $(R, m)$ be a local Noetherian ring, $k \subseteq R$ a subring, and $D \in \mathrm{Der}_k(R)$ a $k$-derivation of $R$. $D$ extends uniquely to a $k$-derivation $\tilde{D} \in \mathrm{Der}_k(\hat{R})$ of the completion $\hat{R}$ of $R$.

<u>Proof</u>: For all $n \in \mathbb{N}$, $a \in m^n$, it holds that $D(a) \in m^{n-1}$. Let $\hat{y} \in \hat{R}$ with $\hat{y} = \lim_{n \in \mathbb{N}} y_n$ where $y_n \in R$ and $y_n - y_{n+k} \in m^n$ for all $n, k \in \mathbb{N}$. Then $D(y_n - y_{n+1}) = D(y_n) - D(y_{n+k}) \in m^{n-1}$ and $\lim_{n \in \mathbb{N}} D(y_n) \in \hat{R}$ is well defined. Define $\tilde{D}(\hat{y}) = \lim_{n \in \mathbb{N}} D(y_n)$ and verify that $\tilde{D}$ is a $k$-derivation of $\hat{R}$.

Let $(R, m, k)$ be a regular local ring of dimension $n$ which contains a coefficient field and let $x_1, \text{—}, x_n$ be a regular system of parameters of $R$. The completion $\hat{R} \cong k[\![x_1, \text{—}, x_n]\!]$ of $R$ is isomorphic to the formal power series ring in $n$ variables over $k$. In the following we assume that $k \subseteq R$ and denote by $\partial/\partial x_i$, $1 \leq i \leq n$, the partial derivations of $k[\![x_1, \text{—}, x_n]\!]$ and the corresponding derivations of $\hat{R}$.

(12.11) <u>Theorem</u>: Assumptions as above. Then:

(a) $\mathrm{Der}_k(\hat{R})$ is a free $\hat{R}$-module with basis $\partial/\partial x_1, \text{—}, \partial/\partial x_n$.

(b) The following conditions are equivalent:

(α) $\partial/\partial x_i \in \mathrm{Der}_k(R)$ for all $1 \leq i \leq n$, that is, for all $1 \leq i \leq n$ $\partial/\partial x_i|_R$ is a $k$-derivation from $R$ into $R$.

($\beta$) There are elements $D_1, \dots, D_n \in \mathrm{Der}_k(R)$ and $a_1, \dots, a_n \in R$ so that $D_i a_j = \delta_{ij}$.

($\gamma$) There are elements $D_1, \dots, D_n \in \mathrm{Der}_k(R)$ and $a_1, \dots, a_n \in R$ so that $\det(D_i a_j)_{i,j=1,\dots,n} \notin m$.

($\delta$) $\mathrm{Der}_k(R)$ is a free $R$-module of rank $n$.

($\varepsilon$) $\mathrm{rank}_R \, \mathrm{Der}_k(R) = n$.

Proof: (a) Identify $\hat{R} = k[[x_1, \dots, x_n]]$ and let $D \in \mathrm{Der}_k(\hat{R})$. Set $Dx_i = y_i$. Then for all $f \in k[[x_1, \dots, x_n]]$

$$(*) \qquad D(f) = \sum_{i=1}^n (\partial f/\partial x_i)\, y_i .$$

In order to prove this formula first prove $(*)$ in the case where $f$ is a polynomial in $x_1, \dots, x_n$. In the general case, let $N \in \mathbb{N}$ and write

$$f = g_N + \sum_{j=1}^q h_j n_j$$

where $g_N \in k[x_1, \dots, x_n]$ a polynomial of total degree $\leq N$, $h_j \in k[[x_1, \dots, x_n]]$ and $n_1, \dots, n_q$ the monomials of total degree $N$. Then verify that

$$D(f) - \sum_{i=1}^n (\partial f/\partial x_i)\, y_i \in (x_1, \dots, x_n)^{N-1}.$$

$(*)$ implies that $\mathrm{Der}_k(\hat{R}) = \sum_{i=1}^n \hat{R}\, \partial/\partial x_i$. Moreover, let $a_1, \dots, a_n \in \hat{R}$ and $D = \sum_{i=1}^n a_i \, \partial/\partial x_i \in \mathrm{Der}_k(\hat{R})$. Obviously, $D = 0$ if and only if $D(x_i) = a_i = 0$ for all $1 \leq i \leq n$. Thus $\mathrm{Der}_k(\hat{R})$ is a free $\hat{R}$-module with basis $\partial/\partial x_1, \dots, \partial/\partial x_n$.

(b) Obviously, $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$ and $(\delta) \Rightarrow (\varepsilon)$.

$(\gamma) \Rightarrow (\delta)$: Let $D_1, \dots, D_n \in \mathrm{Der}_k(R)$ and $a_1, \dots, a_n \in R$ with $\det(D_i a_j) \notin m$. By Lemma (12.10) the $k$-derivations $D_1, \dots, D_n \in \mathrm{Der}_k(R)$ extend uniquely to $k$-derivations of $\mathrm{Der}_k(\hat{R})$ also denoted by $D_1, \dots, D_n$. By Nakayama, $D_1, \dots, D_n$ are a basis of $\mathrm{Der}_k(\hat{R})$ if and only if $D_1 + m\,\mathrm{Der}_k(\hat{R}), \dots, D_n + m\,\mathrm{Der}_k(\hat{R})$ are linearly independent in $\mathrm{Der}_k(\hat{R})/m\,\mathrm{Der}_k(\hat{R})$. Let $y_1, \dots, y_n \in \hat{R}$ with $\sum_{i=1}^n y_i D_i \in m\,\mathrm{Der}_k(\hat{R})$. Then $\sum_{i=1}^n y_i D_i(a_j) \in m\hat{R}$ for all $1 \leq j \leq n$, or equivalently

$$(D_i(a_j)) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \equiv 0 \mod m\hat{R}^n .$$

Since $(D_i(a_j))$ is invertible in $\hat{R}$ (and $\hat{R}/m$) it follows that $y_1, \ldots, y_n \in m\hat{R}$ and $D_1, \ldots, D_n$ is a basis of $\mathrm{Der}_k(\hat{R})$. Since $\mathrm{Der}_k(R) \subseteq \mathrm{Der}_k(\hat{R})$ by Lemma (12.10), the k-derivations $D_1, \ldots, D_n \in \mathrm{Der}_k(R)$ are linearly independent over $R$. Moreover, for every $D \in \mathrm{Der}_k(R)$ there are elements $c_1, \ldots, c_n \in \hat{R}$ with $D = \sum_{i=1}^n c_i D_i$ in $\mathrm{Der}_k(\hat{R})$. Then for all $1 \le j \le n$ $D(a_j) = \sum_{i=1}^n c_i D_i(a_j)$ and

$$(D_i(a_j)) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} D(a_1) \\ \vdots \\ D(a_n) \end{bmatrix} .$$

Since the matrix $(D_i(a_j))$ is invertible in $M_{n\times n}(R)$ it follows that $c_1, \ldots, c_n \in R$ $D_1, \ldots, D_n$ is a basis of $\mathrm{Der}_k(R)$.

$(\varepsilon) \Rightarrow (\alpha)$: Since rank $\mathrm{Der}_k(R) = n$ there are derivations $D_1, \ldots, D_n \in \mathrm{Der}_k(R)$ which are linearly independent over $R$. By Lemma (12.9) there are elements $a_1, \ldots, a_n \in R$ with $\det(D_i(a_j))_{i,j=1,\ldots,n} \neq 0$ and a similar argument as in $(\gamma) \Rightarrow (\delta)$ shows that $D_1, \ldots, D_n$ are linearly independent over $\hat{R}$. Hence there are elements $c_{ij} \in Q(\hat{R})$ so that

$$\partial/\partial x_i = \sum_{j=1}^n c_{ij} D_j .$$

Thus for all $1 \le h \le n$:

$$\delta_{ih} = \partial x_h/\partial x_i = \sum_{j=1}^n c_{ij} D_j(x_h)$$

and the matrix $(c_{ij}) \in M_{n\times n}(Q(\hat{R}))$ is inverse to the matrix $(D_j(x_h)) \in M_{n\times n}(R)$. This implies that $c_{ij} \in Q(R)$ for all $1 \le i,j \le n$. Hence for all $a \in R$ $\partial/\partial x_i(a) \in Q(R) \cap \hat{R} = R$. This shows $(\alpha)$.

(12.12) <u>Proposition</u>: Let $R$ be a regular ring and $P \subseteq R$ a prime ideal of height $r$. The following conditions are equivalent:

(a) There are derivations $D_1, \dots, D_r \in \mathrm{Der}(R)$ and elements $a_1, \dots, a_r \in P$ so that $\det\,(D_i(a_j))_{i,j=1,\dots,r} \notin P$.

(b) For all $Q \in \mathrm{Spec}(R)$ with $Q \subseteq P$, ht $Q = s \leq r$, and $(R/Q)_P$ a regular local ring there are derivations $D_1, \dots, D_s \in \mathrm{Der}(R)$ and elements $b_1, \dots, b_s \in Q$ so that $\det\,(D_i(b_j))_{i,j=1,\dots,s} \notin P$.

Proof: Obviously, (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Let $Q \subseteq R$ be a prime ideal with $Q \subseteq P$, ht $Q = s < r$, and $(R/Q)_P$ a regular local ring. By (a) there are $a_1, \dots, a_r \in P$ so that $\det(D_i(a_j))_{1 \leq i,j \leq r} \notin P$. By (12.6) $a_1, \dots, a_r$ forms a regular system of parameters of $PR_P$. Let $b_1, \dots, b_s \in Q$ be a minimal system of generators of $QR_P$, since $(R/Q)_P$ regular such a system exists. Extend $b_1, \dots, b_s$ to a regular system of parameters of $PR_P$, say $b_1, \dots, b_s, \dots, b_r \in P$. Then there are elements $c_{kj} \in R_P$ so that for all $1 \leq k \leq r$: $a_k = \sum_{j=1}^{r} c_{kj} b_j$ with $\det(c_{kj}) \notin PR_P$. Then for all $1 \leq i,k \leq r$:

$$D_i(a_k) \equiv \sum_{j=1}^{r} c_{kj}\, D_i(b_j) \mod PR_P$$

and thus

$$(D_i(a_k)) \equiv (c_{kj})\,(D_i(b_j)) \mod PM_{r \times r}(R_P).$$

Therefore $\det(D_i(b_j)) \notin P$, in particular,

$$\mathrm{rank}\, J(b_1, \dots, b_r, D_1, \dots, D_r)(P) = r.$$

This implies

$$\mathrm{rank}\, J(b_1, \dots, b_s; D_1, \dots, D_r)(P) = s$$

which proves (a) $\Rightarrow$ (b).

(12.13) Definition: Let $R$ be a regular ring and $P \subseteq R$ a prime ideal. The weak Jacobian condition (WJ) holds in $P$ if condition (a) of Proposition (12.12) is satisfied. We say that $R$ satisfies the weak Jacobian condition if (WJ) holds for every $P \in \mathrm{Spec}(R)$.

If $k \subseteq R$ is a subring we write $(WJ)_k$ if condition (a) of (12.12) is satisfied for $P \in \operatorname{Spec}(R)$ and derivations $D_1, \ldots, D_r \in \operatorname{Der}_k(R)$.

(12.14) Corollary: Let $R$ be a regular ring which satisfies $(WJ)$ and let $P, Q \subseteq R$ be prime ideals with $Q \subseteq P$ and let $Q = s$. The following are equivalent:

(a) $(R/Q)_P$ is a regular local ring.

(b) There are elements $D_1, \ldots, D_s \in \operatorname{Der}(R)$, $b_1, \ldots, b_s \in Q$ with $\det(D_i(b_j)) \notin P$.

Proof: (a) $\Rightarrow$ (b): By Proposition (12.12)

(b) $\Rightarrow$ (a): By Proposition (12.6)

## §4: REGULARITY CRITERIA FOR COMPLETE LOCAL RINGS $R$ WITH $\mathbb{Q} \subseteq R$

(12.15) Theorem: Let $(R, m, k)$ be an $n$-dimensional local Noetherian domain and $\ell \subseteq R$ a subfield of characteristic $0$ with $\operatorname{trdeg}_\ell k = r < \infty$. Then $\operatorname{Der}_\ell(R)$ is isomorphic to a submodule of $R^{n+r}$. In particular, $\operatorname{Der}_\ell(R)$ is a finitely generated $R$-module with

$$\operatorname{rank} \operatorname{Der}_\ell(R) \leq \dim R + \operatorname{trdeg}_\ell k.$$

Proof: Let $u_1, \ldots, u_r$ be a transcendence basis of $k$ over $\ell$ and $x_1, \ldots, x_n \in m$ a system of parameters of $R$. Consider the map $\varphi: \operatorname{Der}_\ell(R) \longrightarrow R^{n+r}$ defined by $\varphi(D) = (Du_1, \ldots, Du_r, Dx_1, \ldots, Dx_n)$. Obviously $\varphi$ is $R$-linear and it remains to show that $\varphi$ is injective. Let $D \in \operatorname{Der}_\ell(R)$ with $\varphi(D) = 0$, that is, $Du_i = 0$ for $1 \leq i \leq r$ and $Dx_j = 0$ for $1 \leq j \leq n$. Extend $D$ to an $\ell$-derivation of the completion $\widehat{R}$. By the proof of (8.18)(b) we may assume that the formal power series ring $S = k[[x_1, \ldots, x_n]]$ is contained in $\widehat{R}$ and that $\widehat{R}$ is a finite $S$-module. Since $k$ is separable algebraic over $\ell(u_1, \ldots, u_r)$ and $Du_i = 0$ for $1 \leq i \leq r$, we obtain that $D|_k = 0$ and hence $D|_S = 0$.

Let $a \in R \subseteq \widehat{R}$. Then there is a nonzero polynomial of minimal degree $f(t) \in S[t]$ with $f(a) = 0$. Since $\operatorname{char} k = 0$, $f'(t) \neq 0$ and thus $f'(a) \neq 0$ since $\deg f' < \deg f$. Moreover, $0 = D(f(a)) = f'(a) D(a)$. By assumption $a \in R$, $D(a) \in R$, and $R$ a domain, hence every nonzero element of $R$ is regular in $\widehat{R}$. This implies $D(a) = 0$ for all $a \in R$.

(12.16) Theorem: Let $(R, m, k)$ be a regular local ring. Assume that $R$ contains a coefficient field of characteristic $0$ also denoted by $k$ $(\subseteq R)$. Then the following conditions are equivalent:

(a) $(WJ)_k$ holds for $m$.

(b) $\operatorname{rank}_R \operatorname{Der}_k(R) = \dim R$

(c) $(WJ)_k$ holds for all $P \in \operatorname{Spec}(R)$

Furthermore, if these conditions hold, then for every $P \in \operatorname{Spec}(R)$ every $k$-derivation of $\operatorname{Der}_k(R/P)$ is induced by a $k$-derivation of $\operatorname{Der}_k(R)$ and

$$\operatorname{rank}_R \operatorname{Der}_k(R/P) = \dim R/P.$$

Proof: (a) $\Longleftrightarrow$ (b): By (12.15) and (12.9)

(c) $\Longrightarrow$ (a): trivial

(a) $\Longrightarrow$ (c): Let $\widehat{R}$ be the completion of $R$ and $x_1, \ldots, x_n \in R$ a regular system of parameters. Then $\widehat{R} \cong k[\![x_1, \ldots, x_n]\!]$. Since $(WJ)_k$ holds at $m$ in $R$, by Theorem (12.11) the partial derivations $\partial/\partial x_i$ restrict to $k$-derivations of $R$ and form a basis of $\operatorname{Der}_k(R)$ (see the proof of (12.11)). Let $P \subseteq R$ be a prime ideal and $\nu : R \longrightarrow R/P$ the natural map. Note that a $k$-derivation $D' \in \operatorname{Der}_k(R/P)$ is induced by the $k$-derivation $D \in \operatorname{Der}_k(R)$ if the following diagram:

$$\begin{array}{ccc} R & \xrightarrow{D} & R \\ \nu \downarrow & & \downarrow \nu \\ R/P & \xrightarrow{D'} & R/P \end{array}$$

commutes. Let $D' \in \operatorname{Der}_k(R/P)$. Then $D'\nu \in \operatorname{Der}_k(R, R/P)$ and $D'\nu$ extends uniquely to a $k$-derivation $D'\nu \in \operatorname{Der}_k(\widehat{R}, \widehat{R}/P\widehat{R})$. Thus $D'\nu$ is uniquely determined by its values on $x_1, \ldots, x_n$. Let $b_1, \ldots, b_n \in R$ with $D'\nu(x_i) = \nu(b_i) = b_i + P$. Set $D = \sum_{i=1}^n b_i \, \partial/\partial x_i \in \operatorname{Der}_k(R)$. Then $\nu D(x_i) = \nu(b_i) = D'\nu(x_i)$ and $D'$ is induced by $D$. Moreover, $\operatorname{Der}_k(R, R/P)$ is a free $R/P$-module with basis $\nu\,\partial/\partial x_i$, $1 \le i \le n$. $\operatorname{Der}_k(R/P)$ can be identified with the submodule

$$N = \{ \delta \in \operatorname{Der}_k(R, R/P) \mid \delta(p) = 0 \text{ for all } p \in P \}.$$

Assume that ht $P = r$ and $P = (p_1, \ldots, p_t)$. Then

$\sum_{i=1}^n a_i \partial/\partial x_i \in \operatorname{Der}_k(R/P) = N \Longleftrightarrow \sum_{i=1}^n a_i \, \partial p_j/\partial x_i = 0$ in $R/P$ for all $1 \le j \le t$.

$$\Longleftrightarrow \quad (\partial p_j/\partial x_i) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0 \quad \text{in } (R/P)^n .$$

Hence

$$\operatorname{rank} \operatorname{Der}_k(R/P) = n - \operatorname{rank} J(p_1, \dots, p_t; \partial/\partial x_1, \dots, \partial/\partial x_n)(P).$$

(Note that the dimension of the null space of the $t\times n$ matrix $(\partial p_j/\partial x_i)$ equals $n - \operatorname{rank}(\partial p_j/\partial x_i)$.)

By (12.6) $\quad \operatorname{rank} J(p_1, \dots, p_t; \partial/\partial x_1, \dots, \partial/\partial x_n)(P) \leq r$

and therefore $\operatorname{rank} \operatorname{Der}_k(R/P) \geq n-r$. On the other hand $\dim R/P = n-r$ and hence by (12.15) $\operatorname{rank} \operatorname{Der}_k(R/P) \leq n-r$. Therefore $\operatorname{rank} \operatorname{Der}_k(R/P) = n-r$ and $\operatorname{rank} J(p_1, \dots, p_t; \partial/\partial x_1, \dots, \partial/\partial x_n)(P) = r$. By (12.12) $(WJ)_k$ holds at $P$.

(12.17) <u>Corollary</u>: Let $k$ be a field of characteristic $0$. The formal power series ring $S = k[[x_1, \dots, x_n]]$ over $k$ satisfies $(WJ)_k$.

<u>Proof</u>: Obviously, $\det(\partial x_j/\partial x_i) = 1 \notin m = (x_1, \dots, x_n)S$ and $(WJ)_k$ holds at $m$. By (12.16) $S$ satisfies $(WJ)_k$.

Let $k$ be a field of characteristic $0$, $S = k[[x_1, \dots, x_n]]$ the formal power series ring over $k$, $P \subseteq S$ a prime ideal and $I \subseteq S$ an ideal with $I \subseteq P$ and let $IS_P = r$. Set $R = (S/I)_P$.

(12.18) <u>Corollary</u>: Assumptions as above. Then the following are equivalent:

(a) $R$ is a regular local ring.

(b) If $I = (a_1, \dots, a_t) \subseteq S$ then

$$\operatorname{rank} J(a_1, \dots, a_t; \partial/\partial x_1, \dots, \partial/\partial x_n)(P) = r.$$

<u>Proof</u>: (a) $\Rightarrow$ (b): First note that by (12.6):

$$\operatorname{rank} \mathcal{J}(a_1, \ldots, a_t; \partial/\partial x_1, \ldots, \partial/\partial x_n)(P) \leq r.$$

Moreover, since $R = (S/I)_P$ regular, there are elements $b_1, \ldots, b_r \in I$ and derivations $D_1, \ldots, D_r \in \operatorname{Der}_k(S)$ with $\det(D_i(b_j))_{1 \leq i,j \leq r} \notin P$. Since $I = (a_1, \ldots, a_t)$ there is an $r \times t$ matrix $\mathcal{U} \in M_{r \times t}(S)$ so that

$$\begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} = \mathcal{U} \begin{bmatrix} a_1 \\ \vdots \\ a_t \end{bmatrix}$$

This implies that

$$(D_i(b_j)) \equiv \mathcal{U}\,(D_i(a_j)) \mod PS^{r \times r}.$$

Moreover, by (12.11) $\operatorname{Der}_k(S)$ is a free $S$-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Hence there is an $r \times n$ matrix $V \in M_{r \times n}(S)$ so that

$$\begin{bmatrix} D_1 \\ \vdots \\ D_r \end{bmatrix} = V \begin{vmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{vmatrix}$$

and therefore

$$(D_i(a_j)) = V\,(\partial a_j/\partial x_i).$$

This shows that

$$(D_i(b_j)) \equiv \mathcal{U}V\,(\partial a_j/\partial x_i) \mod PS^{r \times r}$$

and $\operatorname{rank} \mathcal{J}(a_1, \ldots, a_t; \partial/\partial x_1, \ldots, \partial/\partial x_n)(P) \geq r$.

(b) $\Rightarrow$ (a): By (12.6).

## §5: MORE ON FIELDS OF CHARACTERISTIC $p>0$

(12.19) Definition: Let $k$ be a field.

(a) A subfield $k' \subseteq k$ is called cofinite in $k$ if $[k:k']<\infty$.

(b) A family of subfields $\mathcal{F}=\{k_\alpha\}_{\alpha\in\Gamma}$ of $k$ is called a directed family if for all $\alpha,\beta\in\Gamma$ there is an $\gamma\in\Gamma$ so that $k_\gamma \subseteq k_\alpha \cap k_\beta$.

(12.20) Lemma: Let $k$ be a field of characteristic $p>0$ and $\ell\subseteq k$ a subfield. Then there is a directed family $\mathcal{F}=\{k_\alpha\}_{\alpha\in\Gamma}$ of subfields of $k$ so that

(a) For all $\alpha\in\Gamma$: $\ell\subseteq k_\alpha\subseteq k$ and $[k:k_\alpha]<\infty$, that is, $k_\alpha$ is cofinite in $k$.

(b) $\bigcap_{\alpha\in\Gamma} k_\alpha = \ell(k^p)$.

Proof: Let $B$ be a $p$-basis of $k$ over $\ell$ and $\Gamma$ the set of all finite subsets of $B$. For all $\alpha\in\Gamma$ set $k_\alpha=\ell(k^p, B-\alpha)$. Then $\mathcal{F}=\{k_\alpha\}_{\alpha\in\Gamma}$ is a directed family which satisfies (a) and (b).

(12.21) Proposition: Let $k$ be a field and $\{k_\alpha\}_{\alpha\in\Gamma}$ a directed family of subfields of $k$ with $\ell=\bigcap_{\alpha\in\Gamma} k_\alpha$. Let $V$ be a vector space over $k$ and $v_1,\dots,v_n\in V$ linearly independent over $\ell$. Then there is an $\alpha\in\Gamma$ so that $v_1,\dots,v_n$ are linearly independent over $k_\alpha$.

Proof: For all $\alpha\in\Gamma$ let $q(\alpha)$ be the dimension of the $k_\alpha$-subspace $\sum_{i=1}^n k_\alpha v_i$ of $V$ and $q=\max\{q(\alpha)\mid\alpha\in\Gamma\}$. Then there is a $\beta\in\Gamma$ with $q=q(\beta)$. If $q<n$ we may suppose that $v_1,\dots,v_q$ are linearly independent over $k_\beta$ and that there are elements $c_i\in k_\beta$ with $v_n=\sum_{i=1}^q c_i v_i$. Since $v_1,\dots,v_n$ are linearly independent over $\ell$ there is an $1\le i\le q$ with $c_i\notin\ell$ and thus $c_i\notin k_\gamma$ for some $\gamma\in\Gamma$. Let $\delta\in\Gamma$ with $k_\delta\subseteq k_\beta\cap k_\gamma$ then $v_1,\dots,v_q,v_n$ are linearly independent over $k_\delta$, a contradiction. Hence $q=n$.

(12.22) Proposition: Let $k$ be a field of characteristic $p>0$, $\ell \subseteq k$ a subfield, and $\mathcal{F} = \{k_\alpha\}_{\alpha\in\Gamma}$ a directed family of subfields with $\ell \subseteq k_\alpha \subseteq k$. Then the following conditions are equivalent:

(a) $\bigcap_{\alpha\in\Gamma} k_\alpha(k^p) = \ell(k^p)$

(b) The natural map $\Omega_{k/\ell} \longrightarrow \varprojlim_{\alpha\in\Gamma} \Omega_{k/k_\alpha}$ is injective.

(c) If $u_1, \dots, u_n \in k$ are $p$-independent over $\ell$, then there is an $\alpha\in\Gamma$ with $u_1,\dots,u_n$ $p$-independent over $k_\alpha$.

(d) For every $p$-basis $\mathcal{B}$ of $k$ over $\ell$ and every finite subset $\mathcal{U}$ of $\mathcal{B}$ there is an $\alpha\in\Gamma$ with $\mathcal{U}$ $p$-independent over $k_\alpha$.

(e) There is a $p$-basis $\mathcal{B}$ of $k$ over $\ell$ so that for every finite subset $\mathcal{U}$ of $\mathcal{B}$ there is an $\alpha\in\Gamma$ with $\mathcal{U}$ $p$-independent over $k_\alpha$.

Proof: First note that if $\mathcal{F} = \{k_\alpha\}_{\alpha\in\Gamma}$ is a directed family of subfields of $k$ so is $\mathcal{F}' = \{k_\alpha(k^p)\}_{\alpha\in\Gamma}$.

(a) $\Rightarrow$ (c): Let $\mathcal{U} = \{u_1,\dots,u_n\}$ be $p$-independent over $\ell$. Then the $p^n$ monomials $u_1^{\nu_1}\dots u_n^{\nu_n}$ with $0 \le \nu_i < p$ are linearly independent over $\ell(k^p)$. By (12.21) there is an $\alpha\in\Gamma$ so that $u_1^{\nu_1}\dots u_n^{\nu_n}$, $0\le\nu_i<p$, are linearly independent over $k_\alpha(k^p)$.

(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (e): trivial

(e) $\Rightarrow$ (b): Let $\mathcal{B}$ be a $p$-basis of $k$ over $\ell$ and assume that for every finite subset $\mathcal{U} \subseteq \mathcal{B}$ there is an $\alpha\in\Gamma$ with $\mathcal{U}$ $p$-independent over $k_\alpha$. Let $\omega \in \Omega_{k/\ell}$ with $\omega \neq 0$. Then there are elements $u_1,\dots,u_n \in \mathcal{B}$ and $c_1,\dots,c_n \in \ell(k^p)$ so that $\omega = \sum_{i=1}^n c_i\, d_{k/\ell} u_i$. By assumption there is an $\alpha\in\Gamma$ with $u_1,\dots,u_n$ $p$-independent over $k_\alpha$. Moreover, $u_1,\dots,u_n$ extends to a $p$-basis of $k$ over $k_\alpha$. Hence the image of $\omega$ in $\Omega_{k/k_\alpha}$ is nonzero and the natural map $\Omega_{k/\ell} \longrightarrow \varprojlim_{\alpha\in\Gamma} \Omega_{k/k_\alpha}$ is injective.

(b) $\Rightarrow$ (a): If $a \in k - \ell(k^p)$ then $d_{k/\ell}(a) \neq 0$ in $\Omega_{k/\ell}$. ($a$ extends to a $p$-basis). Hence there is an $\alpha\in\Gamma$ with $d_{k/k_\alpha}(a) \neq 0$ and $a \notin k_\alpha(k^p)$.

(12.23) Proposition: Let $k$ be a subfield of characteristic $p>0$, $\ell \subseteq k$ a subfield, and $\mathcal{F} = \{k_\alpha\}_{\alpha\in\Gamma}$ a directed family of subfields with $\ell \subseteq k_\alpha$ for all $\alpha \in \Gamma$ and $\bigcap_{\alpha\in\Gamma} k_\alpha(k^p) = \ell(k^p)$. Let $k \subseteq K$ be a field extension so that either

(a) $K$ is separable over $k$ or

(b) $K$ is finitely generated over $k$.

Then $\bigcap_{\alpha\in\Gamma} k_\alpha(K^p) = \ell(K^p)$.

Proof: Note that $\mathcal{F}' = \{k_\alpha(k^p)\}_{\alpha\in\Gamma}$ is a directed family of subfields of $k$ and $\mathcal{G} = \{k_\alpha(K^p)\}_{\alpha\in\Gamma}$ is a directed family of subfields of $K$.

(a) Suppose that $k\subseteq K$ is separable and let $\mathcal{B}$ be a p-basis of $k$ over $\ell$, $C$ a p-basis of $K$ over $k$. By (2.14) the sequence

$$0 \longrightarrow \Omega_{k/\ell} \otimes_k K \longrightarrow \Omega_{K/\ell} \longrightarrow \Omega_{K/k} \longrightarrow 0$$

is exact. Hence $\mathcal{B}\cup C$ is a p-basis of $K$ over $\ell$. Let $b_1, \dots, b_m \in \mathcal{B}$ and $c_1,\dots,c_n \in C$ be distinct elements. By (12.22) there is an $\alpha\in\Gamma$ so that $b_1,\dots,b_m$ are p-independent over $k_\alpha$ (as subfield of $k$). Since the sequence $0 \to \Omega_{k/k_\alpha} \otimes_k K \longrightarrow \Omega_{K/k_\alpha} \longrightarrow \Omega_{K/k} \longrightarrow 0$ is exact, $b_1,\dots,b_m,c_1,\dots,c_n$ are p-independent over $k_\alpha$ and the statement follows by (12.22).

(b) If $k\subseteq K$ is finitely generated, $K$ can be obtained from $k$ (in finitely many steps) by a succession of extensions $k' \subseteq k'(x)$ where

(i) $x$ is separable over $k'$

(ii) $x^p = a \in k$.

By (a) it suffices to consider the case where $K = k(x)$ with $x \notin k$ and $x^p = a \in k$.

Case 1: $a \in \ell(K^p)$

Let $\mathcal{B}$ be a p-basis of $k$ over $\ell$. Since $\ell(K^p) = \ell(k^p(a)) = \ell(k^p)$, $\mathcal{B}\cup\{x\}$ is a p-basis of $K$ over $\ell$. Let $\mathcal{U} \subseteq \mathcal{B}\cup\{x\}$ be a finite subset with $\mathcal{U}_0 = \mathcal{U}\cap\mathcal{B} = \{b_1,\dots,b_n\}$. By (12.22) there is an $\alpha\in\Gamma$ so that $\mathcal{U}_0$ is p-independent over $k_\alpha$, that is, the set $\{b_1^{i_1}\cdots b_n^{i_n} \mid 0 \le i_j < p\}$

is linearly independent over $k_\alpha(k^p)$. Then $\{b_1^{i_1}\dots b_n^{i_n}x^i \mid 0\le i_j, i<p\}$ is linearly independent over $k_\alpha(K^p)=k_\alpha(k^p(a))=k_\alpha(k^p)$. The statement follows with (12.22).

Case 2: $a\notin \ell(k^p)$

In this case $\{a\}$ can be extended to a p-basis of $k$ over $\ell$, say $B\cup\{a\}$ where $a\notin B$. Then $B\cup\{x\}$ is a p-basis of $K$ over $\ell$. If $b_1,-,b_n\in B$ then by (12.22) there is an $\alpha\in\Gamma$ with $b_1,-,b_n,a$ p-independent over $k_\alpha$. Then $b_1,-,b_n,x\in K$ are p-independent over $k_\alpha$ (as a subfield of $K$) and by (12.22) $\bigcap_{\alpha\in\Gamma} k_\alpha(K^p)=\ell(K^p)$.

(12.24) Proposition: Let $k$ be a field of characteristic $p>0$ and $\mathcal{F}=\{k_\alpha\}_{\alpha\in\Gamma}$ a directed family of cofinite subfields $k_\alpha\subseteq k$ with $\bigcap_{\alpha\in\Gamma}k_\alpha=k^p$. Let $k\subseteq K$ be a finite field extension. Then there is an $\alpha\in\Gamma$ so that for all subfields $k'\subseteq k_\alpha$ with $[k:k']<\infty$: $\operatorname{rank}_K \Omega_{K/k'}=\operatorname{rank}\Omega_{k/k'}$.

Proof: Consider a chain of intermediate fields $k=\ell_0\subseteq\ell_1\subseteq\dots\subseteq\ell_t=K$ where $\ell_i=\ell_{i-1}(x_i)$ with either $x_i$ separable algebraic over $\ell_{i-1}$ or $x_i^p\in\ell_{i-1}$ and $x_i\notin\ell_{i-1}$. By (12.23) for all $0\le i\le t$: $\bigcap_{\alpha\in\Gamma}k_\alpha(\ell_i^p)=\ell_i^p$. Thus for all $1\le i\le t$ $x_i^p\notin\ell_{i-1}^p$ and there is an $\alpha_i\in\Gamma$ with $x_i^p\notin k_{\alpha_i}(\ell_{i-1}^p)$. Since $\mathcal{F}$ is directed there is an $\alpha\in\Gamma$ with $k_\alpha\subseteq k_{\alpha_i}$ for all $1\le i\le t$. In particular, $x_i^p\notin k_\alpha(\ell_{i-1}^p)$ for all $1\le i\le t$.

The proof is by induction on $i$. Let $\alpha\in\Gamma$ be as above and suppose that $\operatorname{rank}_{\ell_i}\Omega_{\ell_i/k'}=\operatorname{rank}_k\Omega_{k/k'}$ for all $k'\subseteq k_\alpha$ with $[k:k']<\infty$. Let $k'$ be a subfield with $k'\subseteq k_\alpha$, $[k:k']<\infty$, and $\ell_{i+1}=\ell_i(x_{i+1})$. We need to distinguish two cases:

Case 1: $x_i$ is separable over $\ell_i$.

By (2.14) the sequence $0\longrightarrow\Omega_{\ell_i/k'}\otimes_{\ell_i}\ell_{i+1}\longrightarrow\Omega_{\ell_{i+1}/k'}\longrightarrow\Omega_{\ell_{i+1}/\ell_i}\longrightarrow 0$ is exact. Since $\ell_{i+1}$ is separable algebraic over $\ell_i$, $\Omega_{\ell_{i+1}/\ell_i}=0$ and $\Omega_{\ell_i/k'}\otimes_{\ell_i}\ell_{i+1}\cong\Omega_{\ell_{i+1}/k'}$. Thus by induction hypothesis $\operatorname{rank}_{\ell_{i+1}}\Omega_{\ell_{i+1}/k'}=$

$\operatorname{rank}_{\ell_i} \Omega_{\ell_i/k'} = \operatorname{rank}_k \Omega_{k/k'}$.

<u>Case</u> 2: $x_{i+1}^p \in \ell_i$ and $x_{i+1} \notin \ell_i$

Set $a = x_{i+1}^p$, then $\ell_{i+1} \cong \ell_i[t]/(t^p - a)$ and by (1.12) the sequence

$$(t^p-a)/(t^p-a)^2 \xrightarrow{\delta} \Omega_{\ell_i[t]/k'} \otimes_{\ell_i[t]} \ell_{i+1} \longrightarrow \Omega_{\ell_{i+1}/k'} \longrightarrow 0$$

is exact where

$$\Omega_{\ell_i[t]/k'} \otimes_{\ell_i[t]} \ell_{i+1} = (\Omega_{\ell_i/k'} \otimes_{\ell_i} \ell_{i+1}) \oplus \ell_{i+1}\, dt$$

where $d : \ell_i[t] \longrightarrow \Omega_{\ell_i[t]/k'}$ is the universal $k'$-derivation. Since $a = x_{i+1}^p \notin k'(\ell_i^p)$ $\delta(t^p - a) = \delta(-a) \neq 0$ in $\Omega_{\ell_i/k'}$. Thus $\operatorname{rank}_{\ell_{i+1}} \operatorname{im}(\delta) = 1$ and therefore $\operatorname{rank}_{\ell_{i+1}} \Omega_{\ell_{i+1}/k'} = \operatorname{rank}_{\ell_i} \Omega_{\ell_i/k'} = \operatorname{rank}_k \Omega_{k/k'}$.

## §6: NAGATA'S REGULARITY CRITERION

(12.25) Theorem: (Nagata) Let $k$ be a field of characteristic $p>0$, $S=k[\![t_1,\dots,t_m]\!]$ the formal power series ring over $k$, and $P\subseteq S$ a prime ideal. Let $\mathcal{F}=\{k_\alpha\}_{\alpha\in\Gamma}$ be a directed family of cofinite subfields of $k$ with $\bigcap_{\alpha\in\Gamma} k_\alpha = k^p$. Then there is an $\alpha\in\Gamma$ so that for every intermediate field $k^p\subseteq k'\subseteq k_\alpha$ with $[k:k']<\infty$ the following formula holds:

$$\operatorname{rank}_{S/P} \operatorname{Der}_{k'}(S/P) = \dim(S/P) + \operatorname{rank}_k \operatorname{Der}_{k'}(k).$$

Proof: let $T=S/P$ and $L=Q(T)$ its field of quotients. Suppose that $\dim T = n$ and let $x_1,\dots,x_n$ be a system of parameters of $T$. By (10.14) $T$ is finite over the formal power series ring $R=k[\![x_1,\dots,x_n]\!]$. Set $K=Q(R)$, the field of quotients of $R$, and note that $[L:K]<\infty$. Let $k'$ be an intermediate field $k^p\subset k'\subseteq k$ with $[k:k']<\infty$ and let $u_1,\dots,u_r$ be a $p$-basis of $k$ over $k'$. Then $u_1,\dots,u_r,x_1,\dots,x_n$ is a $p$-basis of $R$ over $R'=k'[\![x_1^p,\dots,x_n^p]\!]$, that is, $R^p\subseteq R'$ and $\{u_1^{\nu_1}\cdots u_r^{\nu_r}x_1^{\mu_1}\cdots x_n^{\mu_n} \mid 0\le \nu_i,\mu_j<p\}$ is a basis of the free $R'$-module $R$. $\Omega_{R/R'}$ is generated by $du_1,\dots,du_r,dx_1,\dots,dx_n$ and $\Omega_{K/Q(R')}=\Omega_{R/R'}\otimes_R K$ is a $K$-vector space with basis $du_1,\dots,du_r,dx_1,\dots,dx_n$. Thus $\Omega_{R/R'}$ is a free $R$-module with $\operatorname{rank}_R \Omega_{R/R'} = n+r = \dim R + \operatorname{rank}_k \Omega_{k/k'}$. For all $d\in \operatorname{Der}_{k'}(R)$, $d|_{R'}=0$ and therefore $\operatorname{Der}_{k'}(R)\cong \operatorname{Hom}_R(\Omega_{R/R'},R)$. This shows the assertion if $T=R$ (for all intermediate fields $k^p\subseteq k'\subseteq k$ with $[k:k']<\infty$.

In the general case $T$ is a finite $R'$-module and $\Omega_{T/R'}$ is a finite $T$-module. Since $\operatorname{Der}_{k'}(T)=\operatorname{Der}_{R'}(T)\cong \operatorname{Hom}_T(\Omega_{T/R'},T)$ we obtain:

$$\begin{aligned}
\operatorname{Der}_{k'}(T)\otimes_T L &\cong \operatorname{Hom}_T(\Omega_{T/R'},T)\otimes_T L \\
&\cong \operatorname{Hom}_L(\Omega_{T/R'}\otimes_T L, L) \qquad \text{by 911, Theorem (6.62)} \\
&= \operatorname{Hom}_L(\Omega_{L/Q(R')}, L) \\
&\cong \Omega_{L/Q(R')}
\end{aligned}$$

with $K' = Q(R')$ we have that
$$\operatorname{rank}_T \operatorname{Der}_{k'}(T) = \operatorname{rank}_L \Omega_{L/K'}.$$
Since
$$\operatorname{rank}_K \Omega_{K/K'} = \dim T + \operatorname{rank}_k \operatorname{Der}_{k'}(k)$$
it suffices to find an $\kappa \in \Gamma$ so that for all intermediate fields $k^p \subseteq k' \subseteq k_\kappa$ with $[k:k'] < \infty$ it holds that
$$\operatorname{rank}_L \Omega_{L/K'} = \operatorname{rank}_K \Omega_{K/K'}.$$
In order to apply (12.24) for all $\alpha \in \Gamma$ set $R_\alpha = k_\alpha[\![x_1^p, \ldots, x_n^p]\!]$ and $K_\alpha = Q(R_\alpha)$. Then $\mathcal{G} = \{K_\alpha\}_{\alpha\in\Gamma}$ is a directed family of subfields of $K$. We claim that $\bigcap_{\alpha\in\Gamma} K_\alpha = K^p$. Let $y \in \bigcap_{\alpha\in\Gamma} K_\alpha$ and $\alpha \in \Gamma$. Then there are $a, b \in R_\alpha = k_\alpha[\![x_1^p, \ldots, x_n^p]\!]$ with $y = a/b = ab^{p-1}/b^p$ and $b^p \in R^p \subseteq k^p[\![x_1^p, \ldots, x_n^p]\!] \subseteq R_\beta$ for all $\beta \in \Gamma$. Hence for all $\gamma \in \Gamma$ with $k_\gamma \subseteq k_\alpha$, $yb^p \in R_\alpha \cap K_\gamma$. Since $R_\alpha$ is flat over $R_\gamma$ ($R_\alpha$ is a free $R_\gamma$-module), $R_\alpha \cap K_\gamma = R_\gamma$ and $yb^p \in \bigcap_{\gamma\in\Gamma} R_\gamma = k^p[\![x_1^p, \ldots, x_n^p]\!]$. Thus $\bigcap_{\alpha\in\Gamma} K_\alpha = K^p$ and the assertion follows by (12.24).

Let $k$ be a field of characteristic $p>0$, $S = k[\![x_1, \ldots, x_n]\!]$ the formal power series ring over $k$, and $I, P \subseteq S$ ideals with $I \subseteq P$ and $P$ a prime ideal. Set $A = (S/I)_P$ and suppose that $\operatorname{ht} IS_P = r$ and that $I = (f_1, \ldots, f_t)$ with $f_i \in S$. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be a $p$-basis of $k$ (over $k^p$). For all $\lambda \in \Lambda$ let $D_\lambda \in \operatorname{Der}(k)$ be the derivation defined by $D_\lambda(u_\sigma) = \delta_{\lambda\sigma}$. $D_\lambda$ extends to a derivation $D_\lambda : S \to S$ by defining $D_\lambda(x_i) = 0$ for all $1 \le i \le n$, that is, $D_\lambda$ acts only on the coefficients of a power series of $S$.

(12.26) <u>Theorem</u>: (Nagata's Jacobian criterion) Under assumptions as above the following conditions are equivalent:

(a) $A$ is a regular local ring.

(b) There are finitely many elements $\lambda_1, \ldots, \lambda_m \in \Lambda$ so that

$$\operatorname{rank} J(f_1, \ldots, f_t; D_{\lambda_1}, \ldots, D_{\lambda_m}, \partial/\partial x_1, \ldots, \partial/\partial x_n)(P) = r.$$

Proof: (b) $\Rightarrow$ (a): By (12.6)

(a) $\Rightarrow$ (b): Let $B = \{u_\lambda\}_{\lambda\in\Lambda}$ denote the p-basis of $k$ over its prime field and let $\Gamma$ be the set of all finite subsets of $B$. For all $\alpha \in \Gamma$ let $k_\alpha = k^p(B-\alpha)$. Then $\mathcal{F} = \{k_\alpha\}_{\alpha\in\Gamma}$ is a directed family of subfields of $k$ with $[k:k_\alpha] < \infty$ for all $\alpha\in\Gamma$ and $\bigcap_{\alpha\in\Gamma} k_\alpha = k^p$. By (12,25) there is an $\alpha\in\Gamma$ so that for all intermediate fields $k^p \subseteq k' \subseteq k_\alpha$ with $[k:k'] < \infty$

$$\operatorname{rank}_{S/P} \operatorname{Der}_{k'}(S/P) = \dim(S/P) + \operatorname{rank}_k \operatorname{Der}_{k'}(k)$$

Let $\ell = k_\alpha$, $\alpha = \{u_{\gamma_1}, \ldots, u_{\gamma_s}\}$. Then $\alpha$ is a p-basis of $k$ over $\ell$ and $[k:\ell] = p^s$. In the following set $u_i = u_{\gamma_i}$ for $1\le i\le s$ and $T = \ell[[x_1^p, \ldots, x_n^p]]$. Then $S$ is a finite free T-module with

$$S \cong T[z_1, \ldots, z_s, y_1, \ldots, y_n]/(z_i^p - u_i^p, y_j^p - x_j^p)_{1\le i\le s;\, 1\le j\le n}.$$

With $T[z_1, \ldots, z_s, y_1, \ldots, y_n] = T[\underline{z}, \underline{y}]$ and $J = (z_i^p - u_i^p, y_j^p - x_j^p)_{1\le i\le s;\, 1\le j\le n}$ we have an exact sequence

$$J/J^2 \xrightarrow{\delta} \Omega_{T[\underline{z},\underline{y}]/T} \otimes_{T[\underline{z},\underline{y}]} S \longrightarrow \Omega_{S/T} \longrightarrow 0.$$

Obviously, $\operatorname{im}(\delta) = 0$ and $\Omega_{T[\underline{z},\underline{y}]/T}$ is a free $T[\underline{z},\underline{y}]$-module of rank $s+n$. Thus $\Omega_{S/T}$ is a free S-module with basis $du_1, \ldots, du_s, dx_1, \ldots, dx_n$.

Let $\nu: S \longrightarrow S/P$ be the natural map.

Claim: Every $\ell$-derivation $D' \in \operatorname{Der}_\ell(S/P)$ is induced by an $\ell$-derivation $D \in \operatorname{Der}_\ell(S)$, that is, for all $D' \in \operatorname{Der}_\ell(S/P)$ there is a $D \in \operatorname{Der}_\ell(S)$ so that the diagram

$$\begin{tikzcd} S \arrow[r, "D"] \arrow[d, "\nu"'] & S \arrow[d, "\nu"] \\ S/P \arrow[r, "D'"] & S/P \end{tikzcd}$$

commutes.

Pf. of Claim: Let $D' \in \operatorname{Der}_\ell(S/P)$. $D'$ is uniquely determined by

$D'(\nu(u_1)), \ldots, D'(\nu(u_s)), D'(\nu(x_1)), \ldots, D'(\nu(x_n))$. Let $c_1, \ldots, c_s, b_1, \ldots, b_n \in S$ with $D'(\nu(u_i)) = \nu(c_i)$ and $D'(\nu(x_j)) = \nu(b_j)$ for all $1 \le i \le s$, $1 \le j \le n$ and set

$$D = \sum_{i=1}^{s} c_i D_{\gamma_i} + \sum_{j=1}^{n} b_j \partial/\partial x_j \in \mathrm{Der}_k(S).$$

For any $S$-module $M$ any $k$-derivation $\delta: S \to M$ is determined by $\delta(u_i)$ and $\delta(x_j)$ where $1 \le i \le s$; $1 \le j \le n$. Since for all $1 \le i \le s$, $1 \le j \le n$: $D_{\gamma_i}(u_j) = \delta_{ij}$, $D_{\gamma_i}(x_j) = 0$, $\partial/\partial x_j(u_r) = 0$, and $\partial/\partial x_j(x_r) = \delta_{jr}$, it follows that $\nu D(u_i) = D'(\nu(u_i))$ and $\nu D(x_j) = D'(\nu(x_j))$ for all $1 \le i \le s$, $1 \le j \le n$ and therefore $\nu D = D'\nu$. This proves the claim.

Note that if $D$ and $D'$ are as in the claim then $D(g) \in P$ for all $g \in P$. Similar to the proof of (12.16) we identify $\mathrm{Der}_k(S/P)$ with the following submodule of $\mathrm{Der}_k(S, S/P)$:

$$N = \{\delta \in \mathrm{Der}_k(S, S/P) \mid \delta(g) = 0 \text{ for all } g \in P\}.$$

$\mathrm{Der}_k(S, S/P) \cong \mathrm{Hom}_S(\Omega_{S/T}, S/P) \cong \Omega_{S/T} \otimes_S S/P$ is a free $S/P$-module with basis $\{\nu D_{\gamma_i}, \nu \partial/\partial x_j\}_{1 \le i \le s;\, 1 \le j \le n}$. Therefore

$$\nu\Big(\sum_{i=1}^{s} a_i D_{\gamma_i} + \sum_{j=1}^{n} b_j \partial/\partial x_j\Big) \in \mathrm{Der}_k(S/P) = N \iff$$

$$(*) \qquad \sum_{i=1}^{s} a_i D_{\gamma_i}(g_m) + \sum_{j=1}^{n} b_j \partial/\partial x_j(g_m) \in P \quad \text{for all } 1 \le m \le \ell$$

where $P = (g_1, \ldots, g_\ell)$.

$(*)$ can be written as:

$$J(g_1, \ldots, g_\ell; D_{\gamma_1}, \ldots, D_{\gamma_s}, \partial/\partial x_1, \ldots, \partial/\partial x_n) \begin{bmatrix} a_1 \\ \vdots \\ a_s \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \equiv 0 \mod P.$$

This implies that

$$\mathrm{rank}_{S/P} \mathrm{Der}_k(S/P) = n + s - \mathrm{rank}\, J(g_1, \ldots, g_\ell; D_{\gamma_1}, \ldots, D_{\gamma_s}, \partial/\partial x_1, \ldots, \partial/\partial x_n)(P)$$

On the other hand

$$\begin{aligned}\mathrm{rank}_{S/P} \mathrm{Der}_k(S/P) &= \dim(S/P) + \mathrm{rank}_k \mathrm{Der}_k(k) \\ &= n + s - \mathrm{ht}\, P\end{aligned}$$

and therefore

$$\operatorname{rank} J(g_1,\dots,g_\ell;\, D_{\gamma_1},\dots,D_{\gamma_s},\, \partial/\partial x_1,\dots,\partial/\partial x_n)(P) = \operatorname{ht} P.$$

By assumption $A = (S/I)_P$ is a regular local ring, hence $IS_P = QS_P$ is a prime ideal with $IS_P = (f_1,\dots,f_t)\,S_P$. By (12.12) there are derivations $D_{\lambda_1},\dots,D_{\lambda_q},\, \partial/\partial x_1,\dots,\partial/\partial x_n \in \operatorname{Der}(S)$ so that

$$\operatorname{rank} J(f_1,\dots,f_t;\, D_{\lambda_1},\dots,D_{\lambda_q},\, \partial/\partial x_1,\dots,\partial/\partial x_n)(P) = r = \operatorname{ht} IS_P.$$