# CHAPTER XI: MORE ON THE STRUCTURE OF FORMALLY SMOOTH MORPHISMS

## §1: THE COMPLETE TENSOR PRODUCT

Let $u: (R, m) \longrightarrow (S, n)$ and $v: (R, m) \longrightarrow (T, \omega)$ be local morphisms of local Noetherian rings. For all $i, j \in \mathbb{N}$ we have (not necessarily) injective natural morphisms of $S \otimes_R T$-modules:

$\tau_i: n^i \otimes_R T \longrightarrow S \otimes_R T$ and $\sigma_j: S \otimes_R \omega^j \longrightarrow S \otimes_R T$.

Hence for all $i, j \in \mathbb{N}$ $\operatorname{im}(\tau_i) + \operatorname{im}(\sigma_j)$ is an ideal of $S \otimes_R T$. Let $\mathbb{N}^2$ be partially ordered by $(i,j) \leq (i',j') \iff i \leq i'$ and $j \leq j'$. Then the set $\{ S \otimes_R T / \operatorname{im}(\tau_i) + \operatorname{im}(\sigma_j),\ \nu_{(i,j),(i',j')} \}_{(i,j) \in \mathbb{N}^2}$ is an inverse system where

$$\nu_{(i,j),(i',j')}: S \otimes_R T / \operatorname{im}(\tau_{i'}) + \operatorname{im}(\sigma_{j'}) \longrightarrow S \otimes_R T / \operatorname{im}(\tau_i) + \operatorname{im}(\sigma_j)$$

is the natural map for $(i,j) \leq (i',j')$.

(11.1) Definition: The ring

$$S \hat{\otimes}_R T = \varprojlim_{(i,j) \in \mathbb{N}^2} S \otimes_R T / \operatorname{im}(\tau_i) + \operatorname{im}(\sigma_j)$$

is called the complete tensor product of $S$ and $T$ over $R$.

Define a topology on $S \otimes_R T$ by taking the sets:

$$\{ \operatorname{im}(\tau_i) + \operatorname{im}(\sigma_j) \mid (i,j) \in \mathbb{N}^2 \} = \{ \operatorname{im}(n^i \otimes_R T) + \operatorname{im}(S \otimes_R \omega^j) \mid (i,j) \in \mathbb{N} \}$$

as a basis for the open sets of $S \otimes_R T$. The complete tensor product $S \hat{\otimes}_R T$ is the completion of $S \otimes_R T$ with respect to this topology.

Consider the ideal:

$$\mathfrak{M} = \operatorname{im}(\sigma_1) + \operatorname{im}(\tau_1) = \operatorname{im}(n \otimes_R T) + \operatorname{im}(S \otimes_R \omega) = n(S \otimes_R T) + \omega(S \otimes_R T).$$

Then for all $i \in \mathbb{N}$:

$$\mathfrak{M}^{2i} \subseteq \operatorname{im}(n^i \otimes_R T) + \operatorname{im}(S \otimes_R \omega^i) \subseteq \mathfrak{M}^i$$

and $S \hat{\otimes}_R T$ is the completion of $S \otimes_R T$ with respect to the $\mathfrak{M}$-adic topology.

By the universal property of the tensor product we obtain for all $i,j \in \mathbb{N}$:

$$S \otimes_R T / \mathrm{im}(\tau_i) + \mathrm{im}(\sigma_j) \cong S/n^i \otimes_R T/w^j,$$

in particular,

$$S \otimes_R T/\mathfrak{m} \cong S/n \otimes_R T/w \cong S \hat{\otimes}_R T/\mathfrak{m}.$$

(11.2) <u>Lemma</u>: Let $R$ be a ring and $I \subseteq R$ an ideal. Suppose that $R$ is complete in the $I$-adic topology. Then the following conditions are equivalent:

(a) $R$ is Noetherian

(b) $gr_I(R)$ is Noetherian.

(c) $R/I$ is Noetherian and $I/I^2$ is a finitely generated $R/I$-module.

<u>Proof</u>: Homework

Recall the following theorem from 911:

<u>Theorem</u> A.3: Let $R$ be a ring, $I \subseteq R$ an ideal and $M$ an $R$-module. Suppose that $M$ is $I$-adically ideal-separated. Then the following conditions are equivalent:

(a) $M$ is flat over $R$.

(b) $M/IM$ is flat over $R/I$ and for all $n \in \mathbb{N}$ the natural map:

$$\gamma_n : I^n/I^{n+1} \otimes_{R/I} M/IM \longrightarrow I^nM/I^{n+1}M$$

is bijective.

Note that $\gamma_n$ is always surjective. An $R$-module $M$ is called $I$-adically ideal-separated if for every ideal $J \subseteq R$ the $R$-module $J \otimes_R M$ is separated in the $I$-adic topology.

(11.3) <u>Theorem</u>: Let $u : (R, m) \longrightarrow (S, n)$ and $v : (R, m) \longrightarrow (T, w)$ be local morphisms of local Noetherian rings. Suppose that $T$ is complete in the $\mathfrak{r}$-adic topology and that the residue field $S/\mathfrak{n}$ of $S$ is a finitely generated $R$-module. Then:

(a) $S\hat{\otimes}_R T$ is a complete semilocal Noetherian ring.

(b) The ideal $\mathfrak{n}(S\hat{\otimes}_R T)$ is contained in the Jacobson radical of $S\hat{\otimes}_R T$. Moreover, for all $i \in \mathbb{N}$ with $i > 0$ there is an isomorphism of rings:

$$S\hat{\otimes}_R T/\mathfrak{n}^i(S\hat{\otimes}_R T) \cong S/\mathfrak{n}^i \otimes_R T.$$

(c) If $T$ is flat over $R$, then $S\hat{\otimes}_R T$ is flat over $S$.

<u>Proof</u>: (a) Since $S/\mathfrak{n}$ is a finitely generated $R$-module, the ring $S\otimes_R T/\mathfrak{m} \cong S/\mathfrak{n} \otimes_R T/\mathfrak{r}$ is Artinian. Moreover, $\mathfrak{m}/\mathfrak{m}^2$ is a homomorphic image of $(\mathfrak{n}/\mathfrak{n}^2 \otimes_R T) \oplus (S\otimes_R \mathfrak{r}/\mathfrak{r}^2)$ and hence a finitely generated $S\otimes_R T$-module. By (11.2) $S\hat{\otimes}_R T$ is Noetherian. Since $\mathfrak{m}(S\hat{\otimes}_R T)$ is in the Jacobson radical of $S\hat{\otimes}_R T$ and $S\hat{\otimes}_R T/\mathfrak{m}(S\hat{\otimes}_R T) \cong S\otimes_R T/\mathfrak{m} \cong S/\mathfrak{n} \otimes_R T/\mathfrak{r}$ an Artinian ring, $S\hat{\otimes}_R T$ is a complete semilocal Noetherian ring.

(b) We may write:

$$\begin{aligned} S\hat{\otimes}_R T &= \varprojlim_{(i,j)\in\mathbb{N}^2} S\otimes_R T/\mathrm{im}(\mathfrak{n}^i\otimes_R T) + \mathrm{im}(S\otimes_R \mathfrak{r}^j) \\ &= \varprojlim_{(i,j)\in\mathbb{N}^2} S/\mathfrak{n}^i \otimes_R T/\mathfrak{r}^j \\ &= \varprojlim_i \left(\varprojlim_j S/\mathfrak{n}^i \otimes_R T/\mathfrak{r}^j\right). \end{aligned}$$

For a fixed $i \in \mathbb{N}$ $\varprojlim_j (S/\mathfrak{n}^i \otimes_R T/\mathfrak{r}^j)$ is the completion of the ring $S/\mathfrak{n}^i \otimes_R T$ with respect to the $\mathfrak{r}(S/\mathfrak{n}^i\otimes_R T)$-adic topology. By assumption $S/\mathfrak{n}$ is a finitely generated $R$-module, hence for all $i\in\mathbb{N}$ the ring $S/\mathfrak{n}^i$ is a finitely generated $R$-module and $S/\mathfrak{n}^i \otimes_R T$ is a finitely generated $T$-module. Since $T$ is complete (in the $\mathfrak{r}$-adic topology), so is the ring $S/\mathfrak{n}^i\otimes_R T$ and we obtain

$$S\hat{\otimes}_R T = \varprojlim_i S/\mathfrak{n}^i \otimes_R T.$$

By the universal property of the tensor product

$$\begin{aligned} S/\mathfrak{n}^i \otimes_R T &\cong S\otimes_R T/\mathrm{im}(\mathfrak{n}^i\otimes_R T) \\ &\cong S\otimes_R T/\mathfrak{n}^i(S\otimes_R T) \end{aligned}$$

and $S\hat{\otimes}_R T$ is the $n(S\otimes_R T)$-adic completion of $S\otimes_R T$. In particular, $n(S\hat{\otimes}_R T)$ is contained in the Jacobson radical of $S\hat{\otimes}_R T$ and

$$S\hat{\otimes}_R T/n^i(S\hat{\otimes}_R T) \cong S\otimes_R T/n^i(S\otimes_R T)$$
$$\cong S/n^i \otimes_R T.$$

In order to show (c) we want to apply Theorem A.3 from §11 with $M \cong S\hat{\otimes}_R T$, $R \cong S$, and $I \cong n$. By (b)

$$S\hat{\otimes}_R T/n^i(S\hat{\otimes}_R T) \cong S/n^i \otimes_R T$$

and $S\hat{\otimes}_R T/n^i(S\hat{\otimes}_R T)$ is flat over $S/n^i$, since $T$ is flat over $R$.

In order to show that

$$\gamma_i: n^i/n^{i+1} \otimes_{S/n} S\hat{\otimes}_R T/n(S\hat{\otimes}_R T) \longrightarrow n^i(S\hat{\otimes}_R T)/n^{i+1}(S\hat{\otimes}_R T)$$

is an isomorphism for all $i\in\mathbb{N}$, consider the exact sequence of $S/n^{i+1}$-modules:

$$0 \longrightarrow n^i/n^{i+1} \longrightarrow S/n^{i+1} \longrightarrow S/n^i \longrightarrow 0.$$

Since $S\hat{\otimes}_R T/n^{i+1}(S\hat{\otimes}_R T)$ is flat over $S/n^{i+1}$, the sequence

$$0 \longrightarrow n^i/n^{i+1} \otimes_{S/n^{i+1}} \left(S\hat{\otimes}_R T/n^{i+1}(S\hat{\otimes}_R T)\right) \xrightarrow{\lambda_i} S\hat{\otimes}_R T/n^{i+1}(S\hat{\otimes}_R T)$$

is exact and $\lambda_i$ is injective. Note that

$$n^i/n^{i+1} \otimes_{S/n^{i+1}} \left(S\hat{\otimes}_R T/n^{i+1}(S\hat{\otimes}_R T)\right) \cong \left(n^i/n^{i+1} \otimes_{S/n} S/n\right) \otimes_{S/n^{i+1}} \left(S\hat{\otimes}_R T/n^{i+1}(S\hat{\otimes}_R T)\right)$$
$$\cong n^i/n^{i+1} \otimes_{S/n} \left(S/n \otimes_{S/n^{i+1}} S\hat{\otimes}_R T/n^{i+1}(S\hat{\otimes}_R T)\right)$$
$$\cong n^i/n^{i+1} \otimes_{S/n} \left(S\hat{\otimes}_R T/n(S\hat{\otimes}_R T)\right)$$

and $\gamma_i$ is an isomorphism for all $i\in\mathbb{N}$.

It remains to show that $S\hat{\otimes}_R T$ is $n$-adically ideal separated. Let $J\subseteq S$ be an ideal. Since $S$ is Noetherian, $J\otimes_S(S\hat{\otimes}_R T)$ is a finitely generated $S\hat{\otimes}_R T$-module. By (a) $S\hat{\otimes}_R T$ is Noetherian and by (b) $n(S\hat{\otimes}_R T)$ is contained in the Jacobson radical of $S\hat{\otimes}_R T$. Hence $J\otimes_S(S\hat{\otimes}_R T)$ is separated in the $n(S\hat{\otimes}_R T)$-adic topology and the theorem is proven.

(11.4) <u>Lemma</u>: Let $(D, n, \ell)$ be a discrete valuation ring with maximal ideal $n = xD$ and $(R, m, k)$ a local Noetherian $D$-algebra. If $x$ is $R$-regular, then $R$ is flat over $D$.

Proof: Consider the exact sequence $0 \longrightarrow D \xrightarrow{x} D \longrightarrow \ell \longrightarrow 0$. Tensoring with $R$ over $D$ yields a long exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^D(R,\ell) \longrightarrow R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0.$$

Since $x$ is $R$-regular, $\mathrm{Tor}_1^D(R,\ell)=0$ and since $R$ is Noetherian with $x$ contained in the Jacobson-radical of $R$, $R$ is $xD$-adically ideal-separated. By §11, Theorem A.3 $R$ is flat over $D$.

(11.5) Theorem: Let $(R, m, k)$ be a local Noetherian ring, $(S_0, n_0, \ell)$ a complete regular local ring and a $k$-algebra. Then there is a complete local Noetherian faithfully flat $R$-algebra $(S, n, \ell)$ so that

$$S \otimes_R k \cong S/mS \cong S_0 \quad \text{as } k\text{-algebras}.$$

Proof: We may assume that $R$ is complete and that $k \subseteq S_0$. We need to distinguish two cases:

Case 1: $\ell$ is separable over $k$.

In this case by (8.17) $S_0$ contains a coefficient field $\ell'$ with $k \subseteq \ell'$ and by Cohen's structure theorem $S_0 \cong \ell[\![t_1, -, t_n]\!]$ as $k$-algebras where $t_1, -, t_n$ are variables over $\ell$. By (10.3) there is a complete local ring $(T, v, \ell)$ so that $T$ is faithfully flat over $R$ and $T \otimes_R k \cong \ell$ (in particular, $\ell$ is the residue field of $T$). If $n=0$, that is, $S_0 = \ell$, set $S=T$. If $n \geq 1$, set $S = T[\![t_1, -, t_n]\!]$. Since $T$ is faithfully flat over $R$, so is $S$ and $S \otimes_R k \cong (T/mT)[\![t_1, -, t_n]\!] = \ell[\![t_1, -, t_n]\!] \cong S_0$ as $k$-algebras.

Case 2: $\ell$ is not separable over $k$.

Suppose that $\mathrm{char}\, k = p > 0$ and let $P = \mathbb{F}_p$ denote the prime field of characteristic $p$.

Let $D_0 = \mathbb{Z}_{(p)}$ and $(\hat{D}_0, p\hat{D}_0, P)$ its completion. Since $R$ is complete, the natural map $j: \mathbb{Z} \longrightarrow R$ extends to a morphism of rings $\hat{j}: \hat{D}_0 \longrightarrow R$. By (10.3) there is a complete $p$-ring $(D, pD, k)$ which is flat over $\hat{D}_0$ with

$D \otimes_{\hat{D}_0} P \overset{u_0}{\cong} k$. Moreover, since $k$ is separable over $P$, by (9.15) $D$ is $pD$-smooth over $\hat{D}_0$. By (10.8) there is a local morphism $u: D \longrightarrow R$ which induces the isomorphism $u_0$ on the residue fields.

$S_0$ is a complete regular local ring of equal characteristic and there is a natural morphism $\mu: \hat{D}_0 \longrightarrow S_0$ with $S_0 \otimes_{\hat{D}_0} P \cong S_0$. Moreover, since $\ell$ is separable over $P$, by case 1 there is a faithfully flat complete $\hat{D}_0$-algebra $T$ with $T \otimes_{\hat{D}_0} P \cong S_0$.

Consider the commutative diagram of morphisms of rings

$$\begin{array}{ccccc} D & \xrightarrow{\nu} & k & \xrightarrow{\lambda} & S_0 \\ \uparrow & & & & \uparrow \delta \\ \hat{D}_0 & & \longrightarrow & & T \end{array}$$

where $\delta$ is the natural map. Since $T$ is complete and $D$ is $pD$-smooth over $\hat{D}_0$, $\lambda\nu$ lifts to a local $\hat{D}_0$-algebra morphism $v: D \longrightarrow T$. Moreover,

$$T \otimes_D k \cong T/pT \cong T \otimes_{\hat{D}_0} P \cong S_0 \quad \text{as } k\text{-algebras.}$$

Since $T$ is faithfully flat over $\hat{D}_0$, $p$ is a $T$-regular element and by (11.4) $T$ is faithfully flat over $D$. Thus we have local morphisms $u: (D, pD) \longrightarrow (R, m)$ and $v: (D, pD) \longrightarrow (T, \sigma)$ where $T$ is flat over $D$ and $R/m = k \cong D/pD$ a finitely generated $D$-module. By (11.3) $S = R \hat{\otimes}_D T$ is a complete semilocal Noetherian ring which is flat over $R$.

Moreover,
$$\begin{aligned} S \otimes_R k &\cong R\hat{\otimes}_D T / m(R\hat{\otimes}_D T) \\ &\cong R/m \otimes_D T \\ &\cong T \otimes_D k \cong S_0 \quad \text{as } k\text{-algebras.} \end{aligned}$$

Thus $S$ is a local ring, since by (11.3) $mS = m(R\hat{\otimes}_D T)$ is contained in the Jacobson radical of $R\hat{\otimes}_D T$.

## §2: A LIFTING THEOREM

(11.6) Lemma: Let $R$ and $S$ be rings and $I, J \subseteq S$ ideals. Suppose that there is given a commutative diagram of ring morphisms:

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"'] & S/I \arrow[d, "\nu"] \\ S/J \arrow[r, "\mu"] & S/I+J \end{tikzcd}$$

where $\nu$ and $\mu$ are the natural maps. Then there is a morphism $\delta : R \longrightarrow S/I\cap J$ so that :

$$\beta = \lambda\delta : R \xrightarrow{\delta} S/I\cap J \xrightarrow{\lambda} S/J \quad \text{and}$$
$$\alpha = \rho\delta : R \xrightarrow{\delta} S/I\cap J \xrightarrow{\rho} S/I$$

where $\lambda$ and $\rho$ are the natural maps.

Proof: First note that the natural maps $\lambda : S/I\cap J \longrightarrow S/J$ and $\rho : S/I\cap J \longrightarrow S/I$ induce (by restriction) isomorphisms on the $S$-modules:

$\lambda' : I/I\cap J \xrightarrow{\cong} I+J/J$ and $\rho' : J/I\cap J \xrightarrow{\cong} I+J/I$.

Let $r \in R$ and $x, y \in S$ with $\alpha(r) = x + I \in S/I$ and $\beta(r) = y + J \in S/J$. Then $\lambda(x-y+I\cap J) \in I+J/J$ implying that $x-y+I\cap J \in I/I\cap J$ and $x-y \in I$. Similarly, $\rho(x-y+I\cap J) \in I+J/I$ and therefore $x-y+I\cap J \in J/I\cap J$ and $x-y \in J$. Hence $x-y+I\cap J = 0 + I\cap J$. This shows that for all $r \in R$ there is a unique element $z \in S/I\cap J$ so that $\beta(r) = \lambda(z)$ and $\alpha(r) = \rho(z)$. Define $\delta : R \longrightarrow S/I\cap J$ by $\delta(r) = z$ and verify that $\delta$ is a morphism of rings.

(11.7) Theorem: Let $u : (R, m) \longrightarrow (S, n)$ and $v : (R, m) \longrightarrow (T, t)$ be local morphisms of local Noetherian rings. Suppose that $S$ is $n$-smooth over $R$ and that $T$ is complete. Let $I \subseteq T$ be an ideal and $w_0 : S \longrightarrow T/I$ a local $R$-algebra morphism. Then $w_0$ lifts to a local $R$-algebra morphism $w : S \longrightarrow T$, that is, the diagram

$$\begin{tikzcd} S \arrow[r, "w"] \arrow[dr, "w_0"'] & T \\ & T/I \arrow[u, "\nu"'] \end{tikzcd}$$

commutes where $\nu$ is the natural map.

<u>Proof</u>: Let $\mu_i : T/I \longrightarrow T/I+\varpi^i$ denote the natural map. Since $S$ is $n$-smooth over $R$, for all $i \in \mathbb{N}$ the map $\mu_i w_0 : S \longrightarrow T/I+\varpi^i$ lifts to an $R$-algebra morphism $w_i : S \longrightarrow T/\varpi^i$, that is, the diagram:

$$\begin{tikzcd} S \arrow[r, "\mu_i w_0"] \arrow[dr, "w_i"] & T/I+\varpi^i \\ R \arrow[u] \arrow[r] & T/\varpi^i \arrow[u, "\lambda_i"'] \end{tikzcd}$$

commutes where $\lambda_i$ is the natural map. In general, the $w_i$ may not 'fit together', that is, the diagram:

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[dr, "w_{i+1}"'] & T/\varpi^i \\ & T/\varpi^{i+1} \arrow[u, "\sigma_{i+1}"'] \end{tikzcd}$$

may not commute where $\sigma_{i+1}$ is the natural map. In the following we want to construct for all $i \in \mathbb{N}$ $R$-algebra morphisms $w_i : S \longrightarrow T/\varpi^i$ so that both of the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[dr, "w_{i+1}"'] & T/\varpi^i \\ & T/\varpi^{i+1} \arrow[u, "\sigma_{i+1}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "w_0"] \arrow[d, "w_i"'] & T/I \arrow[d, "\mu_i"] \\ T/\varpi^i \arrow[r, "\lambda_i"] & T/I+\varpi^i. \end{tikzcd}$$

Fix an $i \in I$ and suppose that $w_i : S \longrightarrow T/\varpi^i$ is an $R$-algebra morphism so that the diagram:

$$\begin{tikzcd} S \arrow[r, "w_0"] \arrow[d, "w_i"'] & T/I \arrow[d, "\mu_i"] \\ T/\varpi^i \arrow[r, "\lambda_i"] & T/I+\varpi^i \end{tikzcd}$$

commutes.

We want to construct an $R$-algebra morphism $w_{i+1} : S \longrightarrow T/\varpi^{i+1}$ so that the diagrams

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[dr, "w_{i+1}"'] & T/\varpi^i \\ & T/\varpi^{i+1} \arrow[u, "\sigma_{i+1}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "w_0"] \arrow[d, "w_{i+1}"'] & T/I \arrow[d, "\mu_{i+1}"] \\ T/\varpi^{i+1} \arrow[r, "\lambda_{i+1}"] & T/I+\varpi^{i+1} \end{tikzcd}$$

commute. We first show:

<u>Claim</u>: There is an $R$-algebra morphism $h: S \longrightarrow T/\omega^{i+1}+I\cap\omega^i$ so that the diagrams

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[dr, "h"'] & T/\omega^i \\ & T/\omega^{i+1}+I\cap\omega^i \arrow[u, "\tau"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "w_0"] \arrow[d, "h"'] & T/\omega^i \arrow[d, "\mu_{i+1}"] \\ T/\omega^{i+1}+I\cap\omega^i \arrow[r, "\sigma"] & T/I+\omega^{i+1} \end{tikzcd}$$

commute where $\sigma$ and $\tau$ are the natural maps.

<u>Pf of Claim</u>: Consider the commutative diagram of ring morphisms:

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[d, "\mu_{i+1}w_0"'] & T/\omega^i \arrow[d, "\lambda_i"] \\ T/I+\omega^{i+1} \arrow[r, "\gamma"] & T/I+\omega^i \end{tikzcd}$$

where $\gamma$ is the natural map. Obviously,

$$T/\omega^i \cong (T/\omega^{i+1})/(\omega^i/\omega^{i+1})$$
$$T/I+\omega^{i+1} \cong (T/\omega^{i+1})/(I+\omega^{i+1}/\omega^{i+1})$$
$$T/I+\omega^i \cong (T/\omega^{i+1})/(\omega^i/\omega^{i+1} + I+\omega^{i+1}/\omega^{i+1}).$$

By (11.6) there is an $R$-algebra morphism

$$h: S \longrightarrow (T/\omega^{i+1})/(\omega^i/\omega^{i+1})\cap(I+\omega^{i+1}/\omega^{i+1}) \cong T/\omega^{i+1}+I\cap\omega^i$$

so that the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[dr, "h"'] & T/\omega^i \\ & T/\omega^{i+1}+I\cap\omega^i \arrow[u, "\tau"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "\mu_{i+1}w_0"] \arrow[dr, "h"'] & T/I+\omega^{i+1} \\ & T/\omega^{i+1}+I\cap\omega^i \arrow[u, "\sigma"'] \end{tikzcd}$$

This shows the claim.

In order to finish the proof consider the commutative diagram:

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[dr, dashed, "w_{i+1}"] & T/\omega^{i+1}+I\cap\omega^i \\ R \arrow[u] \arrow[r] & T/\omega^{i+1} \arrow[u, "\rho"'] \end{tikzcd}$$

where $\rho$ is the natural map. Since $S$ is $n$-smooth over $R$, there is an $R$-algebra morphism $w_{i+1}: S \longrightarrow T/\omega^{i+1}$. We obtain the following commutative diagrams:

$$\begin{tikzcd} S \arrow[r, "w_i"] \arrow[d, "w_{i+1}"'] \arrow[dr, "h"] & T/\omega^i \\ T/\omega^{i+1} \arrow[r, "\rho"] & T/\omega^{i+1}+I\cap\omega^i \arrow[u, "\tau"'] \end{tikzcd}.$$

Thus $\sigma_{i+1} w_{i+1} = \tau \varrho w_{i+1} = \tau h = w_i$ and

$$\begin{tikzcd}
S \arrow[r, "w_0"] \arrow[d, "w_{i+1}"'] & T/I \arrow[d, "\mu_{i+1}"] \\
T/\mathfrak{r}^{i+1} \arrow[r, "\lambda_{i+1}"] \arrow[dr, "\varrho"'] & T/\mathfrak{r}^{i+1}+I \\
 & T/\mathfrak{r}^{i+1}+I\cap\mathfrak{r}^i \arrow[u, "\sigma"']
\end{tikzcd}$$

implying that $\lambda_{i+1} w_{i+1} = \sigma \varrho w_{i+1} = \sigma h = \mu_{i+1} w_0$.

This way we can construct a sequence of R-algebra morphisms $w_i : S \longrightarrow T/\mathfrak{r}^i$ so that for all $i \in \mathbb{N}$ the diagram

$$\begin{tikzcd}
S \arrow[r, "w_i"] \arrow[dr, "w_{i+1}"'] & T/\mathfrak{r}^i \\
 & T/\mathfrak{r}^{i+1} \arrow[u, "\sigma_{i+1}"']
\end{tikzcd}$$

commutes and each $w_i$ is a lifting of $\mu_i w_0 : S \longrightarrow T/I+\mathfrak{r}^i$. Since T is complete, there is an R-algebra morphism $w : S \longrightarrow T$ which lifts $w_0$.

## § 3: THE CONVERSE OF THEOREM (9.15)

(11.8) Proposition: Let $u:(R,m,k)\longrightarrow(S,n)$ and $v:(R,m,k)\longrightarrow(T,w)$ be local morphisms of local Noetherian rings. Suppose

(a) $S$ and $T$ are complete

(b) $S$ is $n$-smooth over $R$

(c) $T$ is flat over $R$

(d) There is a $k$-algebra isomorphism $h_0: S\otimes_R k \xrightarrow{\cong} T\otimes_R k$.

Then there is an $R$-algebra isomorphism $h: S \xrightarrow{\cong} T$ which induces $h_0 = h\otimes 1: S\otimes_R k \longrightarrow T\otimes_R k$. In particular, $S$ is flat over $R$ and $T$ is $w$-smooth over $R$.

Proof: Let $\nu: S\longrightarrow S\otimes_R k \cong S/mS$ and $\mu: T\longrightarrow T\otimes_R k\cong T/mT$ be the natural maps. By (11.7) the $R$-algebra morphism $w_0 = h_0\nu: S \xrightarrow{\nu} S\otimes_R k \xrightarrow{h_0} T\otimes_R k \cong T/mT$ lifts to an $R$-algebra morphism $h: S\longrightarrow T$ so that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\;h\;} & T \\ \nu\downarrow & & \downarrow\mu \\ S/mS & \xrightarrow{\;h_0\;} & T/mT \end{array}$$

commutes. By (10.2) $h$ is bijective.

(11.9) Theorem: Let $u:(R,m,k)\longrightarrow(S,n,\ell)$ be a local morphism of local Noetherian rings. If $S$ is $n$-smooth over $R$ then $S$ is flat over $R$ and $S\otimes_R k$ is geometrically regular over $k$.

Proof: By (8.7) $S\otimes_R k$ is $n(S\otimes_R k)$-smooth over $k$. Then by (8.33) $S\otimes_R k$ is geometrically regular over $k$. It remains to show that $S$ is flat over $R$. Since $\hat{S}$ is faithfully flat over $S$ and $n\hat{S}$-smooth over $S$, we may assume that $S$ is complete. Since $S_0 = S\otimes_R k \cong S/mS$ is geometrically regular over $k$, $S_0$ is a regular local ring. By (11.5) there is a complete local Noetherian faithfully flat $R$-algebra $(T, v)$ so that

$$T \otimes_R k \cong T/mT \cong S_0 = S \otimes_R k \quad \text{as } k\text{-algebras}.$$

By (11.8) $T$ and $\hat{S}$ are isomorphic as $R$-algebras. Thus $S$ is flat over $R$.

(11.10) <u>Corollary</u>: Let $u: (R, m, k) \longrightarrow (S, n, \ell)$ be a local morphism of local Noetherian rings and suppose that $S$ is $n$-smooth over $R$. Then $R$ is a regular local ring if and only if $S$ is a regular local ring.

<u>Proof</u>: By (11.9) $S$ is flat over $R$ and $S \otimes_R k \cong S/mS$ is geometrically regular over $k$. In particular, $S/mS$ is a regular local ring. The assertion follows with 911, Theorem (8.63).