CHAPTER XIII: OPENNESS OF LOCI

Let R be a Noetherian ring and M a finitely generated R-module. We are interested in the following subsets of $\mathrm{Spec}(R)$:

$\mathrm{Reg}(R) = \{P \in \mathrm{Spec}(R) \mid R_P \text{ regular}\}$ – the regular locus of R

$\mathrm{Nor}(R) = \{P \in \mathrm{Spec}(R) \mid R_P \text{ normal}\}$ – the normal locus of R

$U_{R_i}(R) = \{P \in \mathrm{Spec}(R) \mid R_P \text{ satisfies } (R_i)\}$ – the (R_i)-locus of R

$U_{S_i}(M) = \{P \in \mathrm{Spec}(R) \mid M_P \text{ satisfies } (S_i)\}$ – the (S_i)-locus of M

$U_{CM}(M) = \{P \in \mathrm{Spec}(R) \mid M_P \text{ Cohen-Macaulay}\}$ – the CM-locus of M.

If R is an arbitrary Noetherian ring these loci may not be open in $\mathrm{Spec}(R)$. In this chapter we want to show openness of these loci provided that R is a complete local Noetherian ring. In the next chapters we extend this result to a larger class of Noetherian rings.

§1: NAGATA'S CRITERION

(13.1) Theorem: Let R be a Noetherian ring and $\mathcal{U} \subseteq \mathrm{Spec}(R)$ a subset. $\mathcal{U}$ is open in $\mathrm{Spec}(R)$ if and only if the following two conditions are satisfied:

(a) For all $P, Q \in \mathrm{Spec}(R)$ with $Q \subseteq P$ and $P \in \mathcal{U}$ it holds that $Q \in \mathcal{U}$.

(b) For all $P \in \mathcal{U}$ a nonempty open subset of $V(P)$ is contained in $\mathcal{U}$.

Proof: If $\mathcal{U} \subseteq \mathrm{Spec}(R)$ is open then conditions (a) and (b) are satisfied.

Conversely, suppose (a) and (b) and set $Y = \mathrm{Spec}(R) - \mathcal{U}$, $\overline{Y}$ the closure of Y in $\mathrm{Spec}(R)$. $\overline{Y}$ is the union of finitely many irreducible components, say $\overline{Y} = V_1 \cup \ldots \cup V_r$ where $V_i = V(P_i)$ for prime ideals $P_i \in \mathrm{Spec}(R)$.

If $P_i \in \mathcal{U}$ for some $1 \leq i \leq r$ then by (b) there is a nonempty open subset $W \subseteq V_i$ with $W \subseteq \mathcal{U}$. Then $Z = V_i - W$ is a closed subset of $\mathrm{Spec}(R)$. We claim that $Y \cap V_i \subseteq Z$. If $Q \in Y \cap V_i$, then $Q_i \in Y$ and

$Q \notin \mathcal{U}$. Since $W \subseteq \mathcal{U}$, $Q \notin W$ and $Y \cap Y_i \subseteq Z$.

Since $\overline{Y} = V_1 \cup \dots \cup V_r$,

$$Y = \overline{Y} \cap Y = (V_1 \cap Y) \cup \dots \cup (V_r \cap Y) \subseteq Z \cup \bigcup_{i \neq j} Y_j$$

contradicting that V_i is an irreducible component of $\overline{Y}$.

Therefore $P_i \notin \mathcal{U}$ for all $1 \leq i \leq r$ and by (a) $V_i \subseteq Y \subseteq \overline{Y}$ for all $1 \leq i \leq r$. Thus $Y = \overline{Y}$ and $\mathcal{U}$ is open in Spec(R).

Let R be a ring and $\mathbb{P}$ a property of local rings. Define

$$\mathbb{P}(R) = \{ P \in \mathrm{Spec}(R) \mid R_P \text{ has property } \mathbb{P} \}.$$

For example, $\mathbb{P}$ can stand for properties: regular, CM, Gorenstein, etc. then $\mathbb{P}(R) = \mathrm{Reg}(R)$, $U_{CM}(R)$, $\mathrm{Gor}(R)$, etc.

(13.2) <u>Definition</u>: We say that the <u>Nagata</u> <u>criterion</u> (NC) holds for a property $\mathbb{P}$ if the following condition is satisfied for every Noetherian ring R:

(NC) If $\mathbb{P}(R/P)$ contains a nonempty open subset of $\mathrm{Spec}(R/P)$ for all $P \in \mathrm{Spec}(R)$ then $\mathbb{P}(R)$ is open in $\mathrm{Spec}(R)$.

(13.3) <u>Theorem</u>: (NC) holds for:

(a) $\mathbb{P} \triangleq$ regular

(b) $\mathbb{P} \triangleq$ CM

(c) $\mathbb{P} \triangleq$ Gorenstein

<u>Proof</u>: Let R be a Noetherian ring with the property that for all $P \in \mathrm{Spec}(R)$ the set $\mathbb{P}(R/P)$ contains a nonempty open subset of $\mathrm{Spec}(R/P)$. We have to show that $\mathbb{P}(R)$ is open in $\mathrm{Spec}(R)$. To do so we want to apply Theorem (13.1). First note that if $P, Q \in \mathrm{Spec}(R)$ are prime ideals with $Q \subseteq P$ and R_P regular (or CM, Gorenstein) then R_Q is regular (or CM, Gorenstein) by 911, (8.29), (8.60), (10.20). Thus in all 3 cases we have to

show that $\mathbb{P}(R)$ satisfies condition (b) of Theorem (13.1).

(a) $\mathbb{P} \hat{=}$ regular

We have to show that for all $P \in \operatorname{Reg}(R)$ there is a nonempty open subset $W \subseteq V(P)$ with $W \subseteq \operatorname{Reg}(R)$. Let $P \in \operatorname{Reg}(R)$ and $x_1, \dots, x_n \in P$ a regular system of parameters of R_P. Then there is an $a \in R-P$ with $PR_a = (x_1, \dots, x_n)R_a$, in particular, $PR_Q = (x_1, \dots, x_n)R_Q$ for all $Q \in D_a$. By assumption $\operatorname{Reg}(R/P)$ contains a nonempty open subset $\tilde{U}$ and $\tilde{U}$ corresponds to an open subset $U \subseteq V(P)$. Since $V(P)$ is irreducible, $U \cap D_a = W \neq \emptyset$. Moreover, for all $Q \in W$, the ring $(R/P)_Q$ is regular and PR_Q is generated by $n = \operatorname{ht} PR_Q$ elements. Hence R_Q is a regular local ring and $W \subseteq \operatorname{Reg}(R)$.

(b) $\mathbb{P} \hat{=}$ CM

We have to show that for all $P \in U_{CM}(R)$ there is a nonempty open subset $W \subseteq V(P)$ with $W \subseteq U_{CM}(R)$.

Let $P \in U_{CM}(R)$ with $\operatorname{ht} P = n$. Then there are $y_1, \dots, y_n \in P$ so that $y_1, \dots, y_n$ is a regular sequence of R_P. For all $1 \le i \le n$ consider a reduced primary decomposition of $(y_1, \dots, y_i)$ and write $(y_1, \dots, y_i) = J_i \cap K_i$ where J_i is the intersection of the primary components of $(y_1, \dots, y_i)$ which are contained in P and K_i is the intersection of primary components not contained in P. Let $a_i \in K_i - P$ for all $1 \le i \le n$ (if $K_i = R$, set $a_i = 1$), and set $a = \prod_{i=1}^{n} a_i$. Since $D_a = \operatorname{Spec}(R_a)$ is open in $\operatorname{Spec}(R)$ with $D_a \cap V(P) \neq \emptyset$ we may replace R by R_a and assume:

(α) $y_1, \dots, y_n$ is an R-sequence

(β) $I = (y_1, \dots, y_n)R$ is P-primary.

Moreover, for all $Q \in V(P)$ we obtain:

$$R_Q \text{ CM} \iff (R/I)_Q \text{ CM}.$$

It remains to show that there is a nonempty open subset $W \subseteq V(P)$ with

(*) $\quad (R/I)_Q$ and $(R/P)_Q$ CM for all $Q \in W$.

Set $\overline{R} = R/I$ and $\overline{P} = P/I$. Since I is P-primary, $\overline{P}$ is the only minimal prime ideal of $\overline{R}$ and there is an $r \in \mathbb{N}$ with $\overline{P}^r = 0$. Consider the filtration $0 = \overline{P}^r \subseteq \overline{P}^{r-1} \subseteq \dots \subseteq \overline{P} \subseteq \overline{R}$ of $\overline{R}$. For all $0 \le i < r$, $\overline{P}^i/\overline{P}^{i+1}$ is a finitely generated $\overline{R}/\overline{P}$-module. Hence for all $0 \le i < r$ there is a $b_i \in R - P$ so that $(\overline{P}^i/\overline{P}^{i+1})\overline{R}_{b_i}$ is a free $\overline{R}_{b_i}$-module. Let $b = \prod_{i=0}^{r-1} b_i$ and $W_1 = D_b \cap V(P)$. By assumption there is a nonempty open subset $W_2 \subseteq V(P)$ with $W_2 \subseteq U_{CM}(R/P)$. Set $W = W_1 \cap W_2$. We claim that for all $Q \in W$ the ring $(R/I)_Q = \overline{R}_Q$ is CM. We show by induction on i that for all $Q \in W$ the ring $(\overline{R}/\overline{P}^i\overline{R})_Q$ is CM.

For $i = 2$ consider the exact sequence

$$0 \longrightarrow (\overline{P}/\overline{P}^2)_Q \longrightarrow (\overline{R}/\overline{P}^2)_Q \longrightarrow (\overline{R}/\overline{P})_Q \longrightarrow 0$$

Since $(\overline{P}/\overline{P}^2)_Q$ is a free $(\overline{R}/\overline{P})_Q$-module, $\operatorname{depth}(\overline{R}/\overline{P})_Q = \operatorname{depth}(\overline{P}/\overline{P}^2)_Q$ and thus by 911, (8.22) $\operatorname{depth}(\overline{R}/\overline{P}^2)_Q \ge \operatorname{depth}(\overline{R}/\overline{P})_Q$. Since $(\overline{R}/\overline{P})_Q$ is CM and $\dim(\overline{R}/\overline{P}^2)_Q = \dim(\overline{R}/\overline{P})_Q$, the ring $(\overline{R}/\overline{P}^2)_Q$ is CM.

For the induction step $i \to i+1$ consider the sequence

$$0 \longrightarrow (\overline{P}^i/\overline{P}^{i+1})_Q \longrightarrow (\overline{R}/\overline{P}^{i+1})_Q \longrightarrow (\overline{R}/\overline{P}^i)_Q \longrightarrow 0$$

and apply a similar argument.

(c) $\mathbb{P} \mathrel{\hat{=}}$ Gorenstein

We have to show that for every $P \in \operatorname{Gor}(R)$ there is a nonempty open subset $W \subseteq V(P)$ with $W \subseteq \operatorname{Gor}(R)$. Let $P \in \operatorname{Gor}(R)$ with $\operatorname{ht} P = n$. Since R_P is CM there are $x_1, -, x_n \in P$ which form an R_P-sequence. As in the proof of (b) there is an $a \in R - P$ so that

(α) $x_1, -, x_n$ is a regular sequence in R_a

(β) The ideal $I = (x_1, -, x_n)R_a$ is PR_a-primary.

Replace R by R_a and assume that $x_1, -, x_n$ is a regular sequence of R with $I = (x_1, -, x_n)$ P-primary. By 911, (10.3) and (10.4) for all $Q \in V(P)$: R_Q is Gorenstein if and only if $(R/I)_Q$ is Gorenstein. We replace R by R/I and assume that P is the unique minimal prime ideal of R.

Since R_P is a 0-dimensional Gorenstein ring by 911, Theorem (10.18):

$$\operatorname{Ext}^1_R(R/P, R) \otimes_R R_P \cong \operatorname{Ext}^1_{R_P}(k(P), R_P) = 0$$

and

$$\operatorname{Hom}_R(R/P, R) \otimes_R R_P \cong \operatorname{Hom}_{R_P}(k(P), R_P) = k(P).$$

Hence there is a $b \in R - P$ so that:

$$\operatorname{Ext}^1_R(R/P, R) \otimes_R R_b = 0 \quad \text{and} \quad \operatorname{Hom}_R(R/P, R) \otimes_R R_b = R_b/PR_b.$$

Since P is the unique minimal prime ideal of R, $P^r = 0$ for some $r \in \mathbb{N}$ and there is a $c \in R - P$ so that $P^i R_c / P^{i+1} R_c$ is a free R_c/PR_c-module for all $0 \le i < r$. Replace R by R_{bc} and assume:

(i) $\operatorname{Ext}^1_R(R/P, R) = 0$ and $\operatorname{Hom}_R(R/P, R) = R/P$

(ii) For all $0 \le i < r$: P^i/P^{i+1} is a free R/P-module.

For all $2 \le i < r$ consider the exact sequence

$$0 \to P^i/P^{i+1} \longrightarrow P/P^{i+1} \longrightarrow P/P^i \longrightarrow 0$$

and the induced long exact sequence:

$$\cdots \longrightarrow \operatorname{Ext}^1_R(P/P^i, R) \longrightarrow \operatorname{Ext}^1_R(P/P^{i+1}, R) \longrightarrow \operatorname{Ext}^1_R(P^i/P^{i+1}, R) \longrightarrow \cdots$$

Since P^i/P^{i+1} is a free R/P-module, $\operatorname{Ext}^1_R(P^i/P^{i+1}, R) \cong \bigoplus \operatorname{Ext}^1_R(R/P, R) = 0$.
Thus by induction on i, $\operatorname{Ext}^1_R(P, R) = 0$.

The exact sequence

$$0 \longrightarrow P \longrightarrow R \longrightarrow R/P \longrightarrow 0$$

yields a long exact sequence:

$$0 = \operatorname{Ext}^i_R(R,R) \longrightarrow \operatorname{Ext}^i_R(P,R) \longrightarrow \operatorname{Ext}^{i+1}_R(R/P, R) \longrightarrow \operatorname{Ext}^{i+1}_R(R,R) = 0$$

showing that $\operatorname{Ext}^2_R(R/P, R) = 0$. Repeat the argument from above to conclude that $\operatorname{Ext}^2_R(P, R) = 0$ which implies that $\operatorname{Ext}^3_R(R/P, R) = 0$, etc.
This shows that $\operatorname{Ext}^i_R(R/P, R) = 0$ for all $i > 0$.

For the last step of the proof notice that if $Q \in V(P)$ and E is an injective R-module then $\operatorname{Hom}_R(R/Q, E) \cong (0 :_E Q)$ is an injective R/Q-module.

<u>Claim</u>: Let $Q \in V(P)$. Then $(R/P)_Q$ is Gorenstein if and only if R_Q is

Gorenstein.

Pf of Claim: Let $0 \to R_Q \to E^\bullet$ be an injective resolution of R_Q (as an R_Q-module). Since $\mathrm{Ext}^i_R(R/P, R) \otimes_R R_Q \cong \mathrm{Ext}^i_{R_Q}((R/P)_Q, R_Q) = 0$ for all $i > 0$, $0 \to (R/P)_Q \cong \mathrm{Hom}_R(R/P, R) \otimes_R R_Q \to \mathrm{Hom}_{R_Q}((R/P)_Q, E^\bullet)$ is an injective resolution of $(R/P)_Q$ (as an $(R/P)_Q$-module). In order to compute $\mathrm{Ext}^i_{R_Q}(k(Q), R_Q)$ and $\mathrm{Ext}^i_{(R/P)_Q}(k(Q), (R/P)_Q)$ we need to consider the complexes $\mathrm{Hom}_{R_Q}(k(Q), E^\bullet)$ and $\mathrm{Hom}_{(R/P)_Q}(k(Q), \mathrm{Hom}_{R_Q}((R/P)_Q, E^\bullet))$. For all $j \in \mathbb{N}$ $\mathrm{Hom}_{R_Q}(k(Q), E^j)$ and $\mathrm{Hom}_{(R/P)_Q}(k(Q), \mathrm{Hom}_{R_Q}((R/P)_Q, E^j))$ are both isomorphic to $(0 :_{E^j(Q)} Q)$ where $E^j(Q)$ is the sum of all copies of $E(R/Q)$ in the direct sum decomposition of E^j (see 911, (7.58) and (7.63)). This implies that $\mathrm{Ext}^i_{R_Q}(k(Q), R_Q) \cong \mathrm{Ext}^i_{(R/P)_Q}(k(Q), (R/P)_Q)$. By 911, Theorem (10.18) R_Q is Gorenstein if and only if $(R/P)_Q$ is Gorenstein.

By the hypothesis in (NC) $\mathrm{Gor}(R/P)$ contains a nonempty open subset W. Then $W \subseteq \mathrm{Gor}(R)$ and by (13.1) $\mathrm{Gor}(R)$ is open in $\mathrm{Spec}(R)$.

(13.4) Theorem: Let S be a homomorphic image of a CM-ring. Then $\mathcal{U}_{CM}(S)$ is open in $\mathrm{Spec}(S)$.

Proof: Let $S = R/I$ where R is a (Noetherian) CM-ring. By (NC) it suffices to show that for all $P \in \mathrm{Spec}(R)$ the set $\mathcal{U}_{CM}(R/P)$ contains a nonempty open subset of $\mathrm{Spec}(R/P)$. Let $P \in \mathrm{Spec}(R)$ with ht $P = n$ and $x_1, \ldots, x_n \in P$ so that $x_1, \ldots, x_n$ form a regular sequence in R_P. Similar to the proof of (13.3)(b) we can assume that

(α) $x_1, \ldots, x_n$ form an R-sequence

(β) The ideal $I = (x_1, \ldots, x_n)$ is P-primary.

Since R is CM, so is R/I and we replace R by R/I. Then P is the unique minimal ideal of R with $P^r = 0$. By replacing R by R_a for a suitable $a \in R - a$ we assume in addition that P^i/P^{i+1} is a free R/P-module for all $0 \le i < r$.

Let $Q \in V(P)$. Then R_Q is CM and we want to show that $(R/P)_Q$ is CM. Set $R' = R_Q$, $P' = PR_Q$ and $Q' = QR_Q$ and consider the exact sequence:

$$(*) \qquad 0 \longrightarrow P'/P'^2 \longrightarrow R'/P'^2 \longrightarrow R'/P' \longrightarrow 0.$$

By 911, Theorem (8.16) for every local Noetherian ring (T, m) and every finitely generated T-module M: depth $M = s$ if and only if $\mathrm{Ext}^i_T(T/m, M) = 0$ for all $i < s$ and $\mathrm{Ext}^s_T(T/m, M) \neq 0$. Applying $\mathrm{Hom}_{R'}(R'/Q', -)$ to $(*)$ yields a long exact sequence:

$$(**) \quad \cdots \longrightarrow \mathrm{Ext}^{i-1}_{R'}(R'/Q', R'/P') \longrightarrow \mathrm{Ext}^i_{R'}(R'/Q', P'/P'^2) \longrightarrow \mathrm{Ext}^i_{R'}(R'/Q', R'/P'^2) \longrightarrow \cdots$$

Moreover, by 911, Theorem (8.22) depth $R'/P'^2 \ge \min(\text{depth } R'/P', \text{depth } P'/P'^2)$ and depth $R'/P' =$ depth P'/P'^2 since P'/P'^2 is a free R'/P'-module. Then by $(**)$ depth $R'/P'^2 =$ depth R'/P'. We now proceed by induction on i. Suppose that depth $R'/P' =$ depth $R'/P'^2 = \ldots =$ depth R'/P'^i and consider the exact sequence

$$0 \longrightarrow P'^i/P'^{i+1} \longrightarrow R'/P'^{i+1} \longrightarrow R'/P'^i \longrightarrow 0.$$

The same argument as above yields that depth $R'/P'^{i+1} =$ depth R'/P'^i.

Since $P'^r = 0$ it follows that depth $R_Q =$ depth $(R/P)_Q$ and $(R/P)_Q$ is CM. Thus there is a nonempty open subset $W \subseteq \mathrm{Spec}(R/P)$ with $W \subseteq U_{CM}(R/P)$ and $U_{CM}(S)$ is open in $\mathrm{Spec}(S)$.

(13.5) <u>Theorem</u>: Let S be the homomorphic image of a Gorenstein ring. Then $\mathrm{Gor}(S)$ is open in $\mathrm{Spec}(S)$.

<u>Proof</u>: Let $S = R/I$ where R is a (Noetherian) Gorenstein ring. By (NC) it suffices to show that for all $P \in \mathrm{Spec}(R)$ the set $\mathrm{Gor}(R/P)$ is open in $\mathrm{Spec}(R/P)$. Let $P \in \mathrm{Spec}(R)$ with ht $P = n$ and $x_1, \ldots, x_n \in P$ so that $x_1, \ldots, x_n$ form a regular sequence of R_P. Similar to the proof of (13.3)(b) we can assume that:

(α) $x_1, \dots, x_n$ form an R-sequence.

(β) The ideal $I = (x_1, \dots, x_n)$ is P-primary.

Since R is Gorenstein, so is R/I and we replace R by R/I and assume that P is the unique minimal prime ideal of R. Replacing R by R_a for a suitable $a \in R - P$ we assume in addition that

(i) P^i/P^{i+1} is a free R/P-module

(ii) $\mathrm{Ext}^1_R(R/P, R) = 0$ and $\mathrm{Hom}_R(R/P, R) = R/P$.

Under these assumptions the proof of (13.3)(c) shows that for all $Q \in V(P)$:

R_Q is Gorenstein $\Longleftrightarrow$ $(R/P)_Q$ is Gorenstein.

Since R is Gorenstein, $(R/P)_Q$ is Gorenstein for all $Q \in V(P)$. Thus there is a nonempty open subset W of $\mathrm{Spec}(R/P)$ with $W \subseteq \mathrm{Gor}(R/P)$.

By (13.1) $\mathrm{Gor}(R/P)$ is open in $\mathrm{Spec}(R/P)$.

Recall that S is an R-algebra essentially of finite type over R if S is the localization of an R-algebra of finite type.

(13.6) <u>Corollary</u>: Let R be a complete local Noetherian ring and S an R-algebra essentially of finite type over R. Then $\mathcal{U}_{CM}(S)$ and $\mathrm{Gor}(S)$ are open in $\mathrm{Spec}(S)$.

<u>Proof</u>: S is a homomorphic image of a regular ring.

(13.7) <u>Corollary</u>: Let K be a field and S a K-algebra essentially of finite type over K. Then $\mathcal{U}_{CM}(S)$ and $\mathrm{Gor}(S)$ are open in $\mathrm{Spec}(S)$.

(13.8) <u>Remark</u>: Theorems (13.4) and (13.5) are not true for $\mathbb{P} \cong$ regular. There is an example of a regular local ring R and a prime ideal $P \subseteq R$ so that $\mathrm{Reg}(R/P)$ is not open in $\mathrm{Spec}(R/P)$.

§2: THE REGULAR LOCUS

(13.9) Lemma: Let $K \subseteq L$ be a finitely generated field extension. Then there is a finite purely inseparable field extension $K \subseteq K'$ so that $L(K')$ is separable over K'!

Proof: We may assume that $\operatorname{char} K = p > 0$ and set $E = K^{p^{-\infty}}$.
Claim: $E \otimes_K L$ is a local Noetherian ring of dimension 0.
Pf of claim: Since $E \otimes_K L$ is the localization of a finitely generated E-algebra, $E \otimes_K L$ is a Noetherian ring. Let $\omega = \sum_{i=1}^{r} a_i \otimes l_i \in E \otimes_K L$. Then there is an $e \in \mathbb{N}$ so that with $q = p^e$: $a_i^q \in K$ for all $1 \leq i \leq r$ and $\omega^p = \sum_{i=1}^{r} a_i^q \otimes l_i^q = 1 \otimes \sum_{i=1}^{r} a_i^q l_i^q$. Thus either $\omega^p = 0$ or ω^p is a unit in $E \otimes_K L$. This shows that the nilradical of $E \otimes_K L$ is maximal and $E \otimes_K L$ is a local Noetherian ring of dimension 0.

Let $m \subseteq E \otimes_K L$ be the maximal ideal and let
$$x_j = \sum_{i=1}^{t} a_{ij} \otimes l_{ij} \quad , \quad 1 \leq j \leq s$$
be generators of m. Set $K' = K(a_{ij})_{1 \leq i \leq t;\, 1 \leq j \leq s}$. Obviously, K' is a finite purely inseparable extension of K. Moreover, $K'^{p^{-\infty}} = K^{p^{-\infty}} = E$ and $E \otimes_{K'} L(K')$ is a local Noetherian ring of dimension. Let $\varphi: E \otimes_K L \longrightarrow E \otimes_{K'} L(K')$ be the ring morphism defined by $\varphi(a \otimes b) = a \otimes b$. φ is surjective with $m = \operatorname{nil}(E \otimes_K L) \subseteq \ker \varphi$. Thus $E \otimes_{K'} L(K')$ is a field and by (1.31) and (1.33) $L(K')$ is separable over K'!

(13.10) Definition: Let R be a Noetherian ring.
(a) R is called Reg-0 if $\operatorname{Reg}(R)$ contains a nonempty open subset of $\operatorname{Spec}(R)$.
(b) R is called Reg-1 if $\operatorname{Reg}(R)$ is open in $\operatorname{Spec}(R)$.
(c) R is called Reg-2 if every finitely generated R-algebra is Reg-1.

The aim is to show that complete local Noetherian rings are Reg $-$ 2.

(13.11) <u>Theorem</u>: Let R be a Noetherian ring. The following conditions are equivalent.

(a) R is Reg -2, that is, every finitely generated R-algebra is Reg -1.

(b) Every finite R-algebra is Reg -1.

(c) For all $P \in \mathrm{Spec}(R)$ and all finite purely inseparable field extensions $k(P) \subseteq K$ there is a finite R-algebra S with:

  (i) $R/P \subseteq S \subseteq K$

  (ii) S is Reg -0

  (iii) $Q(S) = K$.

<u>Proof</u>: Obviously, (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): Let S be a finitely generated R-algebra. Since (NC) holds for $\mathbb{P} \hat{=}$ regular we have to show that for all $Q \in \mathrm{Spec}(S)$ the ring S/Q is Reg -0. Thus it is to show: if S is a domain and a finitely generated R-algebra then S is Reg -0. Let $P \in \mathrm{Spec}(R)$ with $R/P \subseteq S$. Since condition (c) is also satisfied by all rings R/P with $P \in \mathrm{Spec}(R)$ we may assume that $R \subseteq S$ where R and S are domains.

Apply condition (c) to the finite purely inseparable field extension $K = Q(R) \subseteq K$. Then there is a finite R-algebra $R \subseteq R' \subseteq K$ which is Reg -0. Notice that there is an element $b \in R - (0)$ with $R_b = R'_b$ and that $\mathrm{Spec}(R)$ is irreducible. Thus R is Reg -0 and there is an $a \in R$ with R_a a regular ring. Obviously, it is enough to show that S_a is Reg -0 and we may assume that R is a regular ring and S a finitely generated R-algebra with $R \subseteq S$. In addition we assume that R and S are domains. Set $K = Q(R)$ and $L = Q(S)$.

<u>Case 1</u>: L is separable over K.

Since L is finitely generated over K there is a separating transcendence basis $t_1, -, t_n \in S$ of L over K. (See the proof of (1.31) to conclude that S contains a separating transcendence basis of L over K). Set $R' = R[t_1, -, t_n]$ and $K' = Q(R') = K(t_1, -, t_n)$. R' is a regular ring. Moreover, there is a vector space basis $w_1, -, w_t \in S$ of L over K'.

<u>Claim</u> 1: There is an $f \in R' - (0)$ so that S_f is a finite free R'_f-module.

<u>Pf</u> of Claim 1: Since S is finitely generated over R, S is finitely generated over R', say $S = R'[x_1, -, x_r]$. Moreover, each x_i is algebraic over $K' = Q(R')$ and satisfies a nontrivial equation:

$$\sum_{j=0}^{n_i} a_{ij} x_i^j = 0$$

with $a_{ij} \in R'$ and $a_{in_i} \neq 0$. Set $f_1 = \prod_{i=1}^{r} a_{in_i} \neq 0$. Then S_{f_1} is finite over R'_{f_1} and we extend $\omega_1, -, \omega_t$ to a system of generators of the R'_{f_1}-module S_{f_1}, say $\omega_1, -, \omega_t, \sigma_1, -, \sigma_s$. Since $\omega_1, -, \omega_t$ is a basis of L over K' there is an element $f_2 \in R' - (0)$ with $f_2 \sigma_i \in \sum_{j=1}^{t} R'_{f_1} \omega_j$ for all $1 \le i \le s$. Then with $f = f_1 f_2$ the ring S_f is a finite free R'_f-module and Claim 1 is proven.

Since R'_f is a regular ring, we replace R by R'_f and S by S_f and assume that S is a finite free R-module and a domain and that the field extension $Q(R) = K \subseteq L = Q(S)$ is separable algebraic. Let $\omega_1, -, \omega_t$ be a basis of S over R. Consider

$$d = \det(\operatorname{tr}_{L/K}(\omega_i \omega_j))_{1 \le i, j \le t}.$$

Then $d \in R$ and $d \neq 0$ since L is separable over K.

<u>Claim</u> 2: S_d is a regular ring.

<u>Pf</u> of Claim 2: Let $Q \in \operatorname{Spec}(S)$ with $d \notin Q$ and $P = Q \cap R$. Then $S_P = \sum_{i=1}^{t} R_P \omega_i$ is a free R_P-module and

$$\overline{S_P} = S \otimes_R k(P) = \sum_{i=1}^{t} k(P) \overline{\omega_i}$$

is a $k(P)$-vector space of dimension t. Moreover,

$$\overline{d} = \det(\operatorname{tr}_{\overline{S}/k(P)}(\overline{\omega_i}\,\overline{\omega_j})) \neq 0 \quad \text{in } k(P).$$

Then by Bourbaki, Algebra, Chap. V, §8, Prop. 1, $\overline{S}_P$ is étale over $k(P)$. Hence by (3.3) $\overline{S}_P$ is a product of fields and by 911, Theorem (8.63)

S_Q is a regular local ring.

This shows that $\mathrm{Reg}(S)$ contains a nonempty open subset.

Case 2: L is not separable over K.

By Lemma (13.9) there is a finite purely inseparable field extension $K \subseteq K'$ so that $L(K')$ is separable over K'. By assumption (c) there is a finite R-algebra R' so that R' is Reg-0 and $Q(R') = K'$. Set $S' = S[R'] \subseteq L(K')$. By case 1 S' is Reg-0.

Claim 3: S is Reg-0.

Pf of claim 3: Let $w_1, -, w_t \in S'$ with $L(K') = L(w_1, -, w_t)$. Since S' is a finite S-module there are elements $\mu_1, -, \mu_r \in S'$ so that

$$S' = \Sigma_{i=1}^{t} S w_i + \Sigma_{j=1}^{r} S \mu_j .$$

Hence there is an element $g \in S-(0)$ so that

$$S'_g = \Sigma_{i=1}^{t} S_g w_i$$

and S'_g is a finite free S_g-module. Let $Q \in \mathrm{Spec}(S')$ with $g \notin Q$ and set $P = Q \cap S$. The local extension $S_P \longrightarrow S'_Q$ is faithfully flat and by 911, Theorem (8.63) S_P is regular if S_Q is. In particular, S is Reg-0 since S' is Reg-0.

(13.12) Proposition: Let (R, m, k) be a complete local Noetherian ring with $\mathbb{Q} \subseteq R$. Then R is Reg-2.

Proof: By (13.11) it suffices to show: For all $P \in \mathrm{Spec}(R)$ the ring R/P is Reg-0. Hence we may assume that (R, m, k) is a complete local Noetherian domain which contains the rational numbers. By Cohen's structure theorems (8.17) and (8.19) R contains a coefficient field and $R \cong k[[x_1, -, x_n]]/\mathcal{Q}$. Set $S = k[[x_1, -, x_n]]$ and $n = (x_1, -, x_n)$ and suppose that ht $Q = r$. By Theorem (12.11) $(WJ)_k$ holds for $n \subseteq S$. Since S contains a field of characteristic 0, by Theorem (12.16) $(WJ)_k$ holds for every prime ideal $P \subseteq S$. Moreover, $\mathrm{Der}_k(S)$ is a free S-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and by (12.14) for all $P \subseteq S$ with $Q \subseteq P$ the following conditions are equivalent:

(a) $(S/Q)_P$ is regular

(b) $\operatorname{rank} J(g_1, \ldots, g_r; \partial/\partial x_1, \ldots, \partial/\partial x_n)(P) = r$

where $Q = (g_1, \ldots, g_s)$. Let $J \subseteq S$ be the ideal generated by Q and the $r \times r$-minors of the Jacobian matrix $(\partial g_i/\partial x_j)$. Then by (12.14)

$$(S/Q)_P \text{ is regular} \iff J \nsubseteq P.$$

$\operatorname{Reg}(S/Q)$ is open in $\operatorname{Spec}(S/Q)$ and R is Reg-2.

(13.13) <u>Proposition</u>: Let R be a complete local Noetherian ring of equal characteristic $p > 0$. Then R is Reg-2.

<u>Proof</u>: By (13.11) we have to show:

Let $P \subseteq R$ be a prime ideal and $k(P) \subseteq L$ a finite purely inseparable field extension. Then there is a finite R-algebra S with

(i) $R/P \subseteq S \subseteq L$

(ii) S is Reg-0

(iii) $Q(S) = L$.

Let $P \in \operatorname{Spec}(R)$ and $k(P) \subseteq L$ a finite purely inseparable field extension. Replace R by R/P and assume that $L = k(P)(x_1, \ldots, x_m)$ with $[L : k(P)] = q = p^r$ for some $r \in \mathbb{N}$ and $L^q \subseteq k(P)$. We may assume that $x_1, \ldots, x_m$ are integral over R and put $S = R[x_1, \ldots, x_m]$. Obviously, S satisfies conditions (i) and (iii).

<u>Claim</u>: S is a complete local ring.

<u>Pf of Claim</u>: Since S is a finite R-module, S is complete with respect to the Jacobson radical of S. It remains to show that S is local. Consider the set

$$n = \{y \in S \mid y^q \in m \subseteq R\}.$$

Obviously, n is an ideal of S. If $x \in S - n$, then $x^q \notin m$ and x^q is a unit of R. Thus n is the maximal ideal of S and S is local.

Then $S \cong k[[y_1, \ldots, y_n]]/Q$ where $k = S/n$. Set $T = k[[y_1, \ldots, y_n]]$ and assume that ht $Q = r$ with $Q = (g_1, \ldots, g_t)$. Moreover, let $\{u_\gamma\}_{\gamma \in \Gamma}$ be a p-basis of k over k^p and let $D_\gamma \in \mathrm{Der}(T)$ be the derivations defined by $D_\gamma(u_\lambda) = \delta_{\gamma\lambda}$ and $D_\gamma(y_i) = 0$ for all $1 \le i \le n$. If $P \subseteq T$ is a prime ideal with $Q \subseteq P$ then by (12.26) the following conditions are equivalent:

(a) $(T/Q)_P$ is regular

(b) There are $\lambda_1, \ldots, \lambda_s \in \Gamma$ so that

$$\mathrm{rank}\ J(g_1, \ldots, g_t;\ D_{\lambda_1}, \ldots, D_{\lambda_s}, \partial/\partial y_1, \ldots, \partial/\partial y_n)(P) = r.$$

Let $J \subseteq T$ be the ideal generated by Q and all $r \times r$ minors of matrices of the form: $J(g_1, \ldots, g_t;\ D_{\lambda_1}, \ldots, D_{\lambda_s}, \partial/\partial y_1, \ldots, \partial/\partial y_n)$ where $\lambda_1, \ldots, \lambda_s$ is running through all finite subsets of Γ. Then

$$(T/Q)_P \text{ is regular} \iff J \nsubseteq P$$

and $\mathrm{Reg}(T/Q)$ is open in $\mathrm{Spec}(T/Q)$.

(13.14) <u>Theorem:</u> Let (R, m, k) be a complete local Noetherian ring. Then R is Reg-2.

<u>Proof:</u> By (13.11) we have to show:

For every prime ideal $P \subseteq R$ and every purely inseparable field extension $k(P) \subseteq L$ there is a finite R-algebra S with

(i) $R/P \subseteq S \subseteq L$

(ii) $\mathrm{Reg}(S)$ is Reg-0.

(iii) $Q(S) = L$.

Let $P \subseteq R$ be a prime ideal. If char $k(P) > 0$, then R/P is a complete local Noetherian ring of equal characteristic $p > 0$. The assertion follows with (13.13). If char $k(P) = 0$ and char $k = 0$, R is a Reg-2 ring by (13.12). If char $k(P) = 0$ and char $k = p > 0$, then $L = k(P)$ for all finite purely inseparable field extensions of $k(P)$ and it remains to show:

Let R be a complete local Noetherian domain of unequal characteristic.

Then R is Reg-0.

By Cohen's structure theorem (10.14) there is a complete regular local ring $S \subseteq R$ so that R is a finite module over S. Set $Q(R) = L$ and $Q(S) = K$. Similar to the proof of (13.11) (see Claim 1) there is an element $a \in S - (0)$ so that R_a is a finite free S_a-module with basis $\omega_1, \text{—}, \omega_n \in R$. Consider the discriminant:

$$d = \det\left(\mathrm{tr}_{L/K}(\omega_i \omega_j)\right)_{1 \le i \le r,\, 1 \le j \le r} \in S_a .$$

Since L is separable over K it follows that $d \neq 0$. Let $Q \subseteq R$ be a prime ideal with $ad \notin Q$ and set $P = Q \cap S$. The embedding $S_P \longrightarrow R_Q$ is faithfully flat with S_P a regular local ring. Moreover, with $\overline{R} = R \otimes_S k(P) = (R/PR)_P$ we have that

$$\overline{d} = \det\left(\mathrm{tr}_{\overline{R}/k(P)}(\overline{\omega}_i \overline{\omega}_j)\right) \neq 0 \quad \text{in } k(P)$$

and by Bourbaki, Algebra, Chap. $\underline{\text{V}}$, §8, Prop. 1 the ring $\overline{R}$ is étale over $k(P)$. Thus $\overline{R}$ is a product of fields and by 911, Theorem (8.63) the ring R_Q is regular. This shows that R is Reg-0.