# CHAPTER XIV: CHAIN CONDITIONS

## §1: RATLIFF'S THEOREMS

(14.1) Proposition: Let $R$ be a Noetherian ring and $P \subseteq R$ a prime ideal. There are at most finitely many prime ideals $Q \subseteq R$ so that:

(i) $P \subseteq Q$

(ii) $\mathrm{ht}(Q/P) = 1$

(iii) $\mathrm{ht}\, Q > \mathrm{ht}\, P + 1$.

Proof: Suppose that $P \in \mathrm{Spec}(R)$ with $\mathrm{ht}\, P = n$ and let $a_1, \dots, a_n \in P$ with $\mathrm{ht}(a_1, \dots, a_n) = n$. Set $I = (a_1, \dots, a_n)$ and let $P = P_1, P_2, \dots, P_r$ be the prime ideals of $R$ which are minimal over $I$. Let $\{Q_\lambda\}_{\lambda \in \Lambda}$ be an infinite set of prime ideals with $P \subseteq Q_\lambda$ and $\mathrm{ht}(Q_\lambda/P) = 1$ for all $\lambda \in \Lambda$.

Claim: $\bigcap_{\lambda \in \Lambda} Q_\lambda = P$

Pf of Claim: The ideal $J = \bigcap_{\lambda \in \Lambda} Q_\lambda$ is reduced, thus $J$ is the intersection of finitely many prime ideals, say $J = Q_1 \cap \dots \cap Q_r$. If $J \neq P$, then $P \subsetneq Q_i$ for all $1 \leq i \leq r$. Since $\{Q_\lambda\}_{\lambda \in \Lambda}$ is an infinite set, there is a $\lambda \in \Lambda$ with $Q_\lambda \neq Q_i$ for all $1 \leq i \leq r$. On the other hand, since $J \subseteq Q_\lambda$, there is a $1 \leq j \leq r$ with $Q_j \subsetneq Q_\lambda$ and therefore $P \subsetneq Q_j \subsetneq Q_\lambda$ contradicting $\mathrm{ht}(Q_\lambda/P) = 1$. This shows the claim.

Since $\bigcap_{\lambda \in \Lambda} Q_\lambda = P$ and $P_i \nsubseteq P$ for all $2 \leq i \leq r$, it follows that $P_2 \cap \dots \cap P_r \nsubseteq \bigcap_{\lambda \in \Lambda} Q_\lambda$. Thus there is a $\lambda \in \Lambda$ with $P_2 \cap \dots \cap P_r \nsubseteq Q_\lambda$ and $P$ is the only minimal prime of $I$ which is contained in $Q_\lambda$. For every $b \in Q_\lambda - P$ the prime ideal $Q_\lambda$ is minimal over $I + bR = (a_1, \dots, a_n, b)$. By 910, (4.55) $\mathrm{ht}\, Q_\lambda \leq n+1$. This shows the assertion.

(14.2) Theorem: (Ratliff's weak existence theorem) Let $R$ be a Noetherian ring and

$P, Q \subseteq R$ prime ideals with $P \subseteq Q$, $\mathrm{ht}\, P = h$, and $\mathrm{ht}(Q/P) = d > 1$. Then there are infinitely many prime ideals $W \in \mathrm{Spec}(R)$ with $P \subseteq W \subseteq Q$, $\mathrm{ht}\, W = h+1$, and $\mathrm{ht}(Q/W) = d-1$.

Proof: Let $P \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_d \subsetneq Q$ be a strictly increasing chain of prime ideals. If $W \in \mathrm{Spec}(R)$ with $P \subsetneq W \subsetneq P_2$ then $\mathrm{ht}(Q/W) = d-1$. (note if $\mathrm{ht}(Q/W) > d-1$, then $\mathrm{ht}(Q/P) > d$). By 910, (4.56) there are infinitely many prime ideals $W \in \mathrm{Spec}(R)$ with $P \subsetneq W \subsetneq P_2$ and $\mathrm{ht}(W/P) = 1$. By (14.1) for all but finitely many of those prime ideals $\mathrm{ht}\, W = h+1$. Hence there are infinitely many prime ideals $W \subseteq R$ with $P \subsetneq W \subsetneq P_2$, $\mathrm{ht}\, W = h+1$, and $\mathrm{ht}(Q/W) = d-1$.

(14.3) Lemma: Let $R$ be a Noetherian ring, $P \subseteq R$ a prime ideal with $\mathrm{ht}\, P = h \geq 2$ and $v_1, \ldots, v_r \in P$ with $\mathrm{ht}(v_1, \ldots, v_r) = r$. Then $\mathrm{ht}\, P/(v_1, \ldots, v_r) = h - r$.

Proof: By Krull's principal ideal theorem all minimal prime ideals over $I = (v_1, \ldots, v_r)$ have height $r$. If $\mathrm{ht}\, P/I = t < h-r$, then there are elements $v_{r+1}, \ldots, v_{r+t} \in P$ with $\mathrm{ht}\, I + (v_{r+1}, \ldots, v_{r+t})/I = t$. Then $P$ is a minimal prime ideal over $(v_1, \ldots, v_{r+t})$ and by Krull's principal ideal theorem $\mathrm{ht}\, P \leq r+t < h$, a contradiction.

(14.4) Lemma: Let $R$ be a Noetherian ring, $P \subseteq R$ a prime ideal with $\mathrm{ht}\, P = h > 1$, and $u \in P$ an element with $\mathrm{ht}(uR) = 1$. Then there are infinitely many prime ideals $Q \subseteq P$ with $u \notin Q$ and $\mathrm{ht}\, Q = h-1$.

Proof: Let $q_1, \ldots, q_t$ be the minimal prime ideals of $R$. Consider finitely many height one prime ideals $Q_1, \ldots, Q_r$ with $u \notin Q_i$ and $Q_i \subseteq P$ for all $1 \leq i \leq r$. Let $W_1, \ldots, W_s$ be the minimal prime ideals containing $uR$. Note that by Krull's principal ideal theorem $\mathrm{ht}\, W_j = 1$ for all $1 \leq j \leq s$. Since $\mathrm{ht}\, P > 1$, it follows that $P \nsubseteq \bigcup_{i=1}^{t} q_i \cup \bigcup_{j=1}^{r} Q_j \cup \bigcup_{k=1}^{s} W_k = \Delta$. Let $v \in P$ with $v \notin \Delta$.

Then ht $(vR) = 1$ and ht $((u,v)R) = 2$. We proceed by induction on $h$:

Let $h = 2$ and suppose $\{Q_1, -, Q_r\}$ is the set of prime ideals $Q_i \subseteq P$ with $u \notin Q_i$ and ht $Q_i = 1$. (Note that possibly $r = 0$). Let $Q_{r+1}, -, Q_{r+n}$ be the prime ideals which are minimal over $vR$. Since $vR \subseteq P$, there is a $Q_{r+k} \subseteq P$ for some $1 \leq k \leq n$. Then ht $Q_{r+k} = 1$ and $u \notin Q_{r+k}$, a contradiction.

If $h > 2$ set $\overline{R} = R/vR$ and $\overline{P} = P/vR$. Then by (14.3) ht $\overline{P} = h-1$. By assumption $u$ is not contained in a minimal prime ideal over $vR$ and therefore ht $(u\overline{R}) = 1$. By induction hypothesis there are infinitely many prime ideals $\overline{Q}_\alpha \subseteq \overline{R}$ with $\overline{Q}_\alpha \subseteq P$, $u \notin \overline{Q}_\alpha$ and ht $\overline{Q}_\alpha = h-2$. Let $Q_\alpha \subseteq R$ be the preimage of $\overline{Q}_\alpha$ in $R$. Then $Q_\alpha \subseteq P$, $u \notin Q_\alpha$, and ht $Q_\alpha = h-1$.

(14.5) <u>Theorem</u>: (Ratliff's strong existence theorem) Let $R$ be a Noetherian ring and $P, Q \subseteq R$ prime ideals with $P \subseteq Q$, ht $P = h > 0$ and ht $(Q/P) = d > 0$. Then for all $i$ with $0 \leq i < d$ the set

$$\Delta_i = \{ W \subseteq \operatorname{Spec}(R) \mid W \subseteq Q,\ \text{ht}\,(Q/W) = d-i,\ \text{ht}\, W = h+i \}$$

is infinite.

<u>Proof</u>: <u>Case 1</u>: $i > 0$

The proof is by induction on $i$. For $i = 1$, by (14.2) there are infinitely many prime ideals $W \subseteq R$ with $P \subseteq W \subseteq Q$ so that ht $W = h+1$ and ht $(Q/W) = d-1$ and $\Delta_1$ is infinite. For the induction step $i-1 \Rightarrow i$ let $W_0 \in \Delta_{i-1}$, that is, $W_0 \subseteq Q$ with ht $W_0 = h+i-1$ and ht $(Q/W_0) = d-i+1 > 1$. Then apply (14.2) to obtain infinitely many prime ideals $W$ with $W_0 \subseteq W \subseteq Q$ and ht $W = h+i$, ht $(Q/W) = d-i$.

<u>Case 2</u>: $i = 0$

We have to show: Let $Q \subseteq R$ be a prime ideal. If there is a prime ideal $P \subseteq Q$ with ht $P = h > 0$ and ht $(Q/P) = d > 0$, then there are infinitely many prime ideals $W \subseteq Q$ with ht $W = h$ and ht $(Q/W) = d$. We may assume that $R = R_Q$ is a local Noetherian ring with maximal ideal $Q$. Moreover, we replace $R$ by $R/P_0$ for a minimal prime ideal $P_0 \subseteq P$ with $\mathrm{ht}(P/P_0) = h$ and assume that $R$ is a local Noetherian ring with maximal ideal $Q$.

Step 1: Let $a_1, \ldots, a_h \in P$ with $\mathrm{ht}(a_1, \ldots, a_h) = h$, then for all $1 \leq j \leq h$ $\mathrm{ht}(a_1, \ldots, a_j) = j$.

Set $I = (a_1, \ldots, a_h)$ and $J = (a_1, \ldots, a_{h-1})$ (if $h = 1$, set $J = (0)$) and consider a shortest primary decomposition of $I$ and $J$:

$$I = I_1 \cap \ldots \cap I_r \quad \text{and} \quad J = J_1 \cap \ldots \cap J_s$$

where $I_i$ is $P_i$-primary with $P_1 = P$ and $J_j$ is $W_j$-primary. Suppose that $W_j \subseteq P$ for $1 \leq j \leq t$ and $W_j \nsubseteq P$ for $t+1 \leq j \leq s$. Then

$$I_2 \cap \ldots \cap I_r \cap J_{t+1} \cap \ldots \cap J_s \nsubseteq P$$

and let $y \in I_2 \cap \ldots \cap I_r \cap J_{t+1} \cap \ldots \cap J_s - P$. If $r = 1$ and $t = s$ let $y \in Q - P$. Then $y \notin W_j$ for $1 \leq j \leq t$ and therefore

$$I : yR = I_1 \quad \text{and} \quad J : yR = J_1 \cap \ldots \cap J_t .$$

With $x_i = a_i/y \in Q(R)$ set $S = R[x_1, \ldots, x_h]$ and consider the following ideals of $S$ : $K = (x_1, \ldots, x_h)S$, $P' = PS + K$, and $Q' = QS + K$.

Step 2: We claim:

($\alpha$) $S/K \cong R/I_1$

($\beta$) $P'$ and $Q'$ are prime ideals of $S$

($\gamma$) $\mathrm{ht}\, Q' = h + d$

Pf: ($\alpha$) Since $S = R + K$ it follows that $S/K \cong R/K \cap R$ and we have to show that $K \cap R = I_1$. If $\delta \in K \cap R$, then

$$\delta = \sum_{1 \leq |i| \leq \nu} r_{(i)} x^{(i)}$$

where $r_{(i)} \in R$ and $x^{(i)} = x_1^{i_1} \ldots x_h^{i_h}$. Multiplying by $y^\nu$ yields

$$y^\nu \delta = \sum_{1 \leq |i| \leq \nu} r_{(i)} y^{\nu - |i|} a^{(i)} \in I (I + yR)^{\nu - 1}.$$

By the choice of $y$ for all $t \in \mathbb{N}$ $I : y^t R = I_1$ and therefore $\delta \in I_1$. Conversely, $y I_1 \subseteq I$ and therefore $I_1 \subseteq K \cap R$. This shows that $K \cap I_1$ and thus $S/K \cong R/I_1$.

($\beta$) Consider the commutative diagram:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\mu"'] & S \arrow[d, "\lambda"] \\ R/I_1 \arrow[r, "\cong"] & S/K \end{tikzcd}$$

where $\mu, \lambda$ are the natural maps. Then

$$Q/I_1 \cong (QS+K)/K = Q'/K \quad \text{and} \quad P/I_1 \cong (PS+K)/K = P'/K$$

and $S/P' \cong R/P$, $S/Q' \cong R/Q$. $P'$ and $Q'$ are prime ideals of $S$.

($\gamma$) By Matsumura, Commutative Ring Theory, Theorem (13.6)

$$\operatorname{ht}(Q'/K) = \operatorname{ht} Q'/(x_1, -, x_h) \geq \operatorname{ht} Q' - h$$

and therefore

$$\operatorname{ht} Q' \leq h + \operatorname{ht}(Q'/K) = h + \operatorname{ht}(Q/I_1).$$

Since $I_1$ is $P$-primary, $\operatorname{ht}(Q/I_1) = \operatorname{ht}(Q/P) = d$ and hence $\operatorname{ht} Q' \leq h+d$. (Note that we only know that $\operatorname{ht} Q \geq h+d$ and $\operatorname{ht} Q \leq d + \#$ of generators of $I_1$ (or $P$), thus by passing to $S$ we have 'replaced' $I_1$ by $K$, an ideal generated by $h$ elements.)

In order to show that $\operatorname{ht} Q' \geq h+d$ note that $R_y = R[y^{-1}] = S[y^{-1}] = S_y$ and that $PR_y = P'S_y$. Therefore $\operatorname{ht} P' = \operatorname{ht} P = h$. Moreover, since $K \subseteq P'$ and $R/I_1 \cong S/K$, we obtain that $\operatorname{ht}(Q'/P') = \operatorname{ht}(Q/P) = d$ implying that $\operatorname{ht} Q' \geq \operatorname{ht} P' + \operatorname{ht}(Q'/P') = h+d$.

Step 3: For all $\nu \in \mathbb{N}$ consider the ideals

$$L_\nu = (x_1, -, x_{h-1}, x_h - y^\nu) S$$

and let $Q'_\nu \subseteq S$ be a minimal prime ideal over $L_\nu$ with $Q'_\nu \subseteq Q'$ and $\operatorname{ht}(Q'/Q'_\nu) = \operatorname{ht}(Q'/L_\nu)$. For all $\nu \in \mathbb{N}$ set $Q_\nu = Q'_\nu \cap R$.

Claim: For all $\nu \in \mathbb{N}$

($\alpha$) $y \notin Q_\nu$ and $\operatorname{ht} Q_\nu = h = \operatorname{ht} Q'_\nu$

($\beta$) $\operatorname{ht}(Q/Q_\nu) = d$.

Pf of claim: Note that $y \notin P'$ since $PR_y = P'S_y \neq S_y$. Moreover, $I_1 \subseteq R$ is $P$-primary with $R/I_1 \cong S/K$, hence $K$ is $P'$-primary and $\operatorname{ht}(K + yS) = \operatorname{ht}(P' + yS) = \operatorname{ht}(x_1, -, x_h, y) = h+1$. $Q'_\nu$ is a minimal prime ideal over $L_\nu$,

$L_\nu$ is generated by $h$ elements and by Krull's principal ideal theorem $\operatorname{ht} Q'_\nu \le h$. Since $K + yS = L_\nu + yS$ and $\operatorname{ht}(L_\nu + yS) = h+1$, it follows that $y \notin Q'_\nu$.
Using $R_y = S_y$ yields that $Q'_\nu = Q'_\nu S_y \cap S$ and $Q_\nu = Q'_\nu \cap R = Q'_\nu S_y \cap R = Q_\nu R_y \cap R$, in particular, $Q'_\nu S_y = Q_\nu R_y$ and $\operatorname{ht} Q_\nu = \operatorname{ht} Q'_\nu \le h$.

In order to show that $\operatorname{ht} Q_\nu \ge h$ notice that $J : yR = J_1 \cap \text{—} \cap J_t$ and hence for all $r \in J : yR$ : $yr \in J = (a_1, \text{—}, a_{h-1})$ and $r \in (x_1, \text{—}, x_{h-1}) \subseteq S$. Thus $(J : yR) S \subseteq (x_1, \text{—}, x_{h-1}) S$ and

$$(J : yR) + (a_h - y^{\nu+1}) R \subseteq L_\nu \cap R \subseteq Q'_\nu \cap R = Q_\nu .$$

Since $J : yR = J_1 \cap \text{—} \cap J_t$ and $J = J_1 \cap \text{—} \cap J_s$ $(s \ge t)$, all minimal prime ideals over $J : yR$ are also minimal prime ideals over $J$ and therefore $\operatorname{ht}(J : yR) \ge h-1$, since $\operatorname{ht} J = h-1$. Furthermore, all associated prime ideals of $(J : yR)$ are contained in $P$ with $a_h \in P$ and $y \notin P$, thus $(a_h - y^{\nu+1})R$ is not contained in a minimal prime ideal of $J : yR$ and therefore

$$\operatorname{ht}\left[ (J : yR) + (a_h - y^{\nu+1}) R \right] \ge h$$

and $\operatorname{ht} Q_\nu = \operatorname{ht} Q'_\nu \ge h$.

$(\beta)$ Obviously, $S = R + L_\nu = R + Q'_\nu$ and thus $S/Q'_\nu \cong R/Q_\nu$ and $Q'/Q'_\nu \cong Q/Q_\nu$ and $\operatorname{ht}(Q/Q_\nu) = \operatorname{ht}(Q'/Q'_\nu) = \operatorname{ht}(Q'/L_\nu)$. Since $\operatorname{ht} Q' = h + d$ and $L_\nu$ is generated by $h$ elements, by Matsumura, Commutative Ring Theory, Theorem (13.6) $\operatorname{ht}(Q/Q_\nu) = \operatorname{ht}(Q'/Q'_\nu) \ge d$. On the other hand

$$d \le \operatorname{ht}(Q/Q_\nu) = \operatorname{ht}(Q'/Q'_\nu) \le \operatorname{ht} Q' - \operatorname{ht} Q'_\nu = d + h - \operatorname{ht} Q'_\nu .$$

By $(\alpha)$ $\operatorname{ht} Q'_\nu = \operatorname{ht} Q_\nu = h$ yielding that $\operatorname{ht}(Q/Q_\nu) = d$.

<u>Step</u> 4: <u>Claim</u>: For all $\nu, \mu \in \mathbb{N}$ with $\nu \ne \mu$: $Q_\nu \ne Q_\mu$.

<u>Pf</u> of Claim: Since $Q_\nu R_y = Q'_\nu S_y \ne R_y = S_y$, it follows that $Q_\nu = Q_\nu R_y \cap R = Q'_\nu S_y \cap R$ and if $Q_\nu = Q_\mu$ then $Q'_\nu = Q'_\mu$. Suppose that $Q_\nu = Q_\mu$ with $\nu < \mu$, then $y^\mu - y^\nu = y^\nu (y^{\mu-\nu} - 1) \in Q'_\nu + Q'_\mu = Q'_\nu$. Since $R$ is local, $y^{\mu-\nu} - 1$ is a unit and therefore $y^\nu \in Q'_\nu$, a contradiction. Hence $Q_\nu \ne Q_\mu$ whenever $\nu \ne \mu$ and $\Delta_0$ is infinite.

Recall that a Noetherian ring $R$ is called catenary if for all prime ideals $P, Q \subseteq R$ with $P \subseteq Q$ all saturated chains of prime ideals $P = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \text{—} \subsetneq P_t = Q$ have the same length.

(14.6) <u>Theorem</u>: A local Noetherian domain $(R, m)$ is catenary if and only if for all prime ideals $P \subseteq R$

$$(*) \qquad \operatorname{ht} P + \dim R/P = \dim R.$$

<u>Proof</u>: If $R$ is catenary, formula $(*)$ holds since $R$ is a domain. Conversely, if $R$ is not catenary there are prime ideals $P, Q \subseteq R$ and saturated chains of prime ideals

(1) $P = P_0 \subsetneq P_1 \subsetneq \text{—} \subsetneq P_t = Q$ and (2) $P = Q_0 \subsetneq \text{—} \subsetneq Q_s = Q$

with $s \neq t$, say $s < t$. If for all $1 \leq i \leq s$ $\operatorname{ht} Q_i = \operatorname{ht} Q_{i-1} + 1$, then $\operatorname{ht} Q = \operatorname{ht} Q_{s-1} + 1 = \operatorname{ht} Q_{s-2} + 2 = \text{—} = \operatorname{ht} P + s$. But by (1) $\operatorname{ht} Q \geq \operatorname{ht} P + t > \operatorname{ht} P + s$, a contradiction. Thus, if $R$ is not catenary, there are prime ideals $P, Q \subseteq R$ with $P \subseteq Q$, $\operatorname{ht}(Q/P) = 1$, and $\operatorname{ht} Q > \operatorname{ht} P + 1$. Suppose that $\operatorname{ht}(m/Q) = d$ and apply (14.5) to $R/P$, $Q/P$, and $m/P$ (with $i = 0$). Then there are infinitely many prime ideals $Q_\lambda \subseteq R$ with $P \subseteq Q_\lambda$ so that $\operatorname{ht}(Q_\lambda/P) = 1$ and $\operatorname{ht}(m/Q_\lambda) = d$. Assume in addition that formula $(*)$ holds. Then

$$\operatorname{ht}(m/Q_\lambda) + \operatorname{ht} Q_\lambda = \dim R = n \quad \text{and} \quad \operatorname{ht}(m/Q) + \operatorname{ht} Q = \dim R = n.$$

Thus for all $\lambda$ : $\operatorname{ht} Q_\lambda = n - d = \operatorname{ht} Q > \operatorname{ht} P + 1$. By (14.1) there are at most finitely many prime ideals $Q' \subseteq R$ with $P \subseteq Q'$, $\operatorname{ht}(Q'/P) = 1$ and $\operatorname{ht} Q' > \operatorname{ht} P + 1$.

## §2: FORMALLY EQUIDIMENSIONAL RINGS

(14.7) Definition: A Noetherian ring $R$ of finite Krull dimension is called equidimensional if $\dim R = \dim R/P$ for all minimal prime ideals $P \subseteq R$.

(14.8) Lemma: Let $(R, m)$ be a local catenary Noetherian ring. If $R$ is equidimensional then for all prime ideals $P, Q \subseteq R$ with $P \subseteq Q$:

$$\operatorname{ht} Q = \operatorname{ht}(Q/P) + \operatorname{ht} P.$$

Proof: Let $P_0 \subseteq R$ be a minimal prime ideal with $P_0 \subseteq P$. Since $R$ is catenary, so is $R/P_0$ and by (14.6)

$$\operatorname{ht}(P/P_0) = \operatorname{ht}(m/P_0) - \operatorname{ht}(m/P) = \dim R - \operatorname{ht}(m/P).$$

This shows that $\operatorname{ht}(P/P_0)$ is independent of the choice of the minimal prime ideal $P_0 \subseteq P$ and $\operatorname{ht} P = \operatorname{ht}(P/P_0)$ for every minimal prime ideal $P_0 \subseteq P$. Similarly, $\operatorname{ht} Q = \operatorname{ht}(Q/P_0)$ for all minimal prime ideals $P_0 \subseteq Q$. Let $P_0 \subseteq R$ be a minimal prime ideal with $P_0 \subseteq P \subseteq Q$. Then by (14.6), since $(R/P_0)_Q$ is catenary:

$$\begin{aligned}\operatorname{ht} Q = \operatorname{ht}(Q/P_0) = \dim((R/P_0)_Q) &= \operatorname{ht}(Q/P) + \operatorname{ht}(P(R/P_0)_Q) \\ &= \operatorname{ht}(Q/P) + \operatorname{ht}(P/P_0) \\ &= \operatorname{ht}(Q/P) + \operatorname{ht} P.\end{aligned}$$

For the next theorem we use the following facts from §11:

Let $\varphi: (R, m) \longrightarrow (S, n)$ be flat local morphism of local Noetherian rings.

(a) For all $P \in \operatorname{Spec}(R)$ there is a $Q \in \operatorname{Spec}(S)$ with $\varphi^{-1}(Q) = P$.

(b) If $P \in \operatorname{Spec}(R)$ and $Q \subseteq S$ a minimal prime ideal over $PS$ then $Q \cap R = P$ and $\operatorname{ht} Q = \operatorname{ht} P$.

(14.9) Theorem: Let $\varphi: (R, m) \longrightarrow (S, n)$ be a local morphism of local Noetherian rings. Suppose that

(i) $S$ is faithfully flat over $R$.

(ii) $S$ is equidimensional and catenary.

Then

(a) $R$ is equidimensional.

(b) For all prime ideals $P \subseteq R$ the ring $S/PS$ is equidimensional.

(c) $R$ is catenary.

Proof: (a) Let $P_0 \subseteq R$ be a minimal prime ideal and $Q_0 \subseteq S$ a prime ideal minimal over $P_0 S$. Then $Q_0$ is minimal in $\operatorname{Spec}(S)$. By assumption (ii) $\dim S = \dim S/Q_0 = \dim S/P_0 S$. The induced morphism $\bar{\varphi}: R/P_0 \longrightarrow S/P_0 S$ is faithfully flat and by §11 (8.63) $\dim S/P_0 S = \dim R/P_0 + \dim S/mS$. Thus $\dim R/P_0 = \dim S - \dim S/mS$ is independent of the choice of $P_0$ and $R$ is equidimensional.

(b) Let $P \subseteq R$ be a prime ideal and $Q \subseteq S$ a prime ideal which is minimal over $PR$. Then ht $P =$ ht $Q$. Since $S$ is catenary and equidimensional for every minimal prime ideal $Q_0 \subseteq Q$ with ht $Q =$ ht$(Q/Q_0)$ by (14.6)

$$\dim S = \dim S/Q_0 = \operatorname{ht}(Q/Q_0) + \dim S/Q$$

and $\dim S/Q = \dim S - \operatorname{ht} Q = \dim S - \operatorname{ht} P$ is independent of the choice of $Q$. $S/PS$ is equidimensional.

(c) We have to show that for every minimal prime ideal $P_0 \subseteq R$ and every prime ideal $P \subseteq R$ with $P_0 \subseteq P$ the formula of (14.6) holds, namely

$$\operatorname{ht}(P/P_0) + \dim R/P = \dim R/P_0 .$$

Note that by (a) $R$ is equidimensional and $\dim R/P_0 = \dim R$.

Let $P_0 \subseteq R$ be a minimal prime ideal. The induced morphism $\bar{\varphi}: R/P_0 \longrightarrow S/P_0 S$ is faithfully flat and by (b) $S/P_0 S$ is catenary and equidimensional. Thus we may replace $R$ by $R/P_0$, $S$ by $S/P_0 S$, and $\varphi$ by $\bar{\varphi}$ and assume that $\varphi: (R,m) \longrightarrow (S,n)$ is a local flat morphism of local Noetherian rings and that $R$ is a domain and $S$ is equidimensional and catenary. We have to show that for every prime ideal $P \subseteq R$ the formula of (14.6) $\mathrm{ht}\,P + \dim R/P = \dim R$ holds.

Let $Q \subseteq S$ be a prime ideal which is minimal over $PS$, then $\mathrm{ht}\,Q = \mathrm{ht}\,P$. Let $Q_0 \subseteq S$ be a minimal prime ideal of $S$ with $\mathrm{ht}\,Q = \mathrm{ht}(Q/Q_0)$. Since $S/Q_0$ is catenary, by (14.6)

$$\dim S/Q_0 = \dim S = \mathrm{ht}\,(Q/Q_0) + \dim (S/Q)$$

and hence, since $S/PS$ and $S$ are equidimensional,

$$\dim S/PS = \dim S/Q = \dim S - \mathrm{ht}\,Q = \dim S - \mathrm{ht}\,P.$$

Since $\varphi : R \to S$ and the induced morphism $\bar{\varphi} : R/P \to S/PS$ are faithfully flat, by 911 (8.63):

$$\dim S = \dim R + \dim S/mS \quad \text{and} \quad \dim S/PS = \dim R/P + \dim S/mS.$$

Thus
$$\begin{aligned} \dim S/PS &= \dim R/P + \dim S/mS \\ &= \dim R + \dim S/mS - \mathrm{ht}\,P \end{aligned}$$

and therefore $\dim R = \dim R/P + \mathrm{ht}\,P$. By (14.6) $R$ is catenary.

(14.10) <u>Corollary</u>: Let $R$ be a local Noetherian ring and $\hat{R}$ its completion. If $\hat{R}$ is equidimensional, $R$ is equidimensional and catenary.

(14.11) <u>Corollary</u>: Let $R$ be a regular local ring, $I \subseteq R$ an ideal, and $S = R/I$. If $S$ is equidimensional then the completion $\hat{S}$ is equidimensional.

<u>Proof</u>: Let $Q_0 \subseteq \hat{S}$ be a minimal prime ideal, $P_0 = Q_0 \cap S$, and let $P \subseteq R$, $Q \subseteq \hat{R}$ be prime ideals with $I \subseteq P$, $I\hat{R} \subseteq Q$ and $P/I = P_0$, $Q/I\hat{R} = Q_0$. Note that $P_0$ is a minimal prime ideal of $S$. Since $R$ is regular, $\hat{R}$ is a regular local ring and a domain. In particular, $\hat{R}$ is equidimensional and catenary. By (14.9) applied to the faithfully flat extension $R \hookrightarrow \hat{R}$ the ring $\hat{R}/P\hat{R}$ is equidimensional. Thus $\dim \hat{S}/Q_0 = \dim \hat{R}/Q = \dim \hat{R}/P\hat{R} = \dim R/P = \dim S/P_0 = \dim S$ since $S$ is equidimensional. Hence $\hat{S}$ is equidimensional.

(14.12) <u>Definition</u>: A local Noetherian ring $R$ is called <u>formally equidimensional</u> if $\widehat{R}$ is equidimensional.

(14.13) <u>Remark</u>: A formally equidimensional local Noetherian ring is equidimensional.

(14.14) <u>Theorem</u>: Let $(R,m)$ be a formally equidimensional local Noetherian ring.

(a) For all $P \in \mathrm{Spec}(R)$ the ring $R_P$ is formally equidimensional.

(b) Let $I \subseteq R$ be an ideal. Then $R/I$ is equidimensional if and only if $R/I$ is formally equidimensional.

(c) Let $S$ be a local ring and an $R$-algebra essentially of finite type. If $S$ is equidimensional then $S$ is formally equidimensional.

(d) $R$ is universally catenary.

<u>Proof</u>: (a) Let $P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(\widehat{R})$ with $Q \cap R = P$. Consider the commutative diagram of local morphisms:

$$\begin{array}{ccc} S = \widehat{R}_Q & \xrightarrow{\varepsilon} & (\widehat{R}_Q)^\wedge = \widehat{S} \\ \sigma \uparrow & & \uparrow \tau \\ R_P & \xrightarrow{\mu} & (R_P)^\wedge \end{array}$$

By 911 Theorem A.3 $\tau$ is faithfully flat. Moreover, $S$ and $\widehat{S}$ are homomorphic images of regular local rings. Since $R$ is formally equidimensional, $S = \widehat{R}_Q$ is equidimensional and by (14.11) $\widehat{S}$ is equidimensional. Since $\widehat{S}$ is also catenary, by (14.9) $(R_P)^\wedge$ is equidimensional.

(b) "$\Rightarrow$": Let $I \subseteq R$ be an ideal with $R/I$ equidimensional and $Q \subseteq \widehat{R}$ a prime ideal with $I\widehat{R} \subseteq Q$ and $Q$ minimal over $I\widehat{R}$. Then $P = R \cap Q$ is minimal over $I$ and $\dim R/I = \dim R/P$. Since $R$ is equidimensional, $\widehat{R}$ is equidimensional and catenary and by (14.9) $\widehat{R}/P\widehat{R}$ is equidimensional. Thus

$$\dim \widehat{R}/Q = \dim \widehat{R}/P\widehat{R} = \dim R/P = \dim R/I = \dim \widehat{R}/I\widehat{R}.$$

$\widehat{R}/I\widehat{R}$ is equidimensional.

"$\Leftarrow$": If $I \subseteq R$ is an ideal with $\hat{R}/I\hat{R}$ equidimensional, then $R/I$ is equidimensional by (14.9) since $\hat{R}$ is catenary.

(c) Let $\overline{S} = (R[t_1, -, t_n]/I)_{n}$ where $t_1, -, t_n$ are variables and $I, n \subseteq R[t_1, -, t_n]$ ideals with $n$ a prime ideal and $I \subseteq n$. By (b) it suffices to show that the localized polynomial ring $S = R[t_1, -, t_n]_n$ is formally equidimensional. The natural map $R[t_1, -, t_n] \longrightarrow \hat{R}[t_1, -, t_n]$ is faithfully flat and there is a prime ideal $w \subseteq \hat{R}[t_1, -, t_n]$ with $w \cap R[t_1, -, t_n] = n$. Set $T = \hat{R}[t_1, -, t_n]_w$ and note that the faithfully flat morphism $\varphi: S \longrightarrow T$ extends to a faithfully flat morphism of the completions $\hat{\varphi}: \hat{S} \longrightarrow \hat{T}$.

<u>Claim:</u> $\hat{T}$ is equidimensional

Pf of claim: With $Q = \hat{R} \cap w$ the natural morphism $\hat{R}_Q \longrightarrow T$ is faithfully flat. Set $K = k(Q) = (\hat{R}/Q)_Q$, then by 911, Theorem (8.63):

$$\dim T = \dim \hat{R}_Q + \dim K[t_1, -, t_n]_w .$$

If $P_0 \subseteq \hat{R}_Q$ is a minimal prime ideal, by (14.8) $ht\, Q = \dim \hat{R}_Q = ht(Q/P_0) = \dim \hat{R}_Q/P_0$. Moreover, since $\hat{R}_Q/P_0 \longrightarrow T/P_0T$ is faithfully flat by 911 (8.63)

$$\dim T/P_0T = \dim \hat{R}_Q/P_0 + \dim K[t_1, -, t_n]_w$$
$$= \dim \hat{R}_Q + \dim K[t_1, -, t_n]_w = \dim T.$$

All minimal prime ideals of $T$ are of the form $P_0T$ where $P_0 \subseteq \hat{R}_Q$ is a minimal prime ideal of $\hat{R}_Q$. This shows that $T$ is equidimensional and hence by (14.11) $T$ is formally equidimensional since $T$ is a homomorphic image of a regular local ring. This shows that $\hat{T}$ is equidimensional.

By (14.9) the ring $\hat{S}$ is equidimensional, that is, $S$ is formally equidimensional.

(d) Let $S$ be a local domain which is essentially of finite type over $R$. By (c) $S$ is formally equidimensional and hence catenary by (14.10). This implies that $R$ is universally catenary.

## §3: UNIVERSALLY CATENARY RINGS

(14.15) Definition: A local Noetherian ring $R$ is called formally catenary if $R/P$ is formally equidimensional for all $P \in \operatorname{Spec}(R)$.

(14.16) Proposition: Let $(R, m)$ be a local Noetherian ring. If $R$ is formally catenary then $R$ is universally catenary.

Proof: $R$ is universally catenary if and only if $R/P_0$ is universally catenary for every minimal prime ideal $P_0 \subseteq R$. Hence we may assume that $R$ is a formally catenary domain. Then $R$ is formally equidimensional and $R$ is universally catenary by (14.14).

(14.17) Remark: Let $(R, m)$ be a local Noetherian ring. Then

$$\begin{tikzcd} R \text{ formally equidimensional} \arrow[r, Rightarrow, "(14.14)", "(1)"'] \arrow[d, Rightarrow, "(14.14)"', "(2)"] & R \text{ universally catenary} \\ R \text{ formally catenary} \arrow[ur, Rightarrow, "(3)", "(14.16)"'] & \end{tikzcd}$$

There are many examples of universally catenary local Noetherian rings which are not equidimensional, for example, if $R$ is a regular local ring and $P, Q \subseteq R$ prime ideals with $P \nsubseteq Q$ and $\operatorname{ht} P < \operatorname{ht} Q$, then $R/P \cap Q$ is not equidimensional. Thus the converse of (1) and (2) is false. In this section we want to show that the converse of (3) holds. This requires some preparation.

(14.18) Proposition: Let $(R, m)$ be a local catenary domain with $\dim R = n$ and $\hat{R}$ its completion. Suppose that there is a minimal prime ideal $Q \subseteq \hat{R}$ so that $1 < \dim \hat{R}/Q = d < n$. For all $1 \leq i \leq d-1$ let $\Delta_i$ denote the set of prime ideal $P \subseteq R$ with

(a) $\operatorname{ht} P = i$

(b) There is a prime ideal $W \subseteq \hat{R}$ which is minimal over $P\hat{R}$ so that $Q \subseteq W$ and

$\dim \hat{R}/W = d-i$.

Then for all $1 \le i \le d-1$ : $\Delta_i \neq \emptyset$.

Proof: The proof is by induction on $i$:

$i=1$: Let $a \in m \subseteq R$ with $a \neq 0$ and $W \subseteq \hat{R}$ a minimal prime ideal over $a\hat{R}+Q$. Then $\mathrm{ht}(W/Q)=1$ and $W \cap R = P \neq 0$. This implies that

$$\hat{\Gamma} = \{W \in \mathrm{Spec}(\hat{R}) \mid Q \subseteq W,\ \mathrm{ht}(W/Q)=1 \text{ and } W\cap R \neq 0\}$$

is a nonempty set with

$$m = \bigcup_{W \in \Gamma} (W \cap R).$$

Since $\mathrm{ht}(m\hat{R}/Q) = \dim \hat{R}/Q = d > 1$, $m\hat{R} \notin \hat{\Gamma}$ and for all $W \in \hat{\Gamma}$, $W \cap R \neq m$. Thus $\hat{\Gamma}$ is an infinite set and so is the set $\Gamma = \{W \cap R \mid W \in \hat{\Gamma}\}$. By (14.1) there are at most finitely many prime ideals $W \subseteq \hat{R}$ with $Q \subseteq W$, $\mathrm{ht}(W/Q)=1$ and $\mathrm{ht}\, W > \mathrm{ht}\, Q+1 = 1$. Hence there are infinitely many $W \in \Gamma$ with $\mathrm{ht}\, W = \mathrm{ht}(W/Q)=1$ and $\mathrm{ht}(W \cap R) = 1$. Moreover, since $\hat{R}/Q$ is catenary

$$\dim \hat{R}/W = \dim \hat{R}/Q - \mathrm{ht}(W/Q) = d-1$$

and $P = W \cap R \in \Delta_1$.

For the induction step $i-1 \Rightarrow i$ let $i>1$ and $P \in \Delta_1$. Set $\overline{R} = R/P$ and let $Q_0 \subseteq \hat{R}$ be a minimal prime ideal over $P\hat{R}$ with $Q \subseteq Q_0$ and $\dim \hat{R}/Q_0 = d-1$. Since $R$ is a catenary domain, $\dim R/P = n-1 = \dim \hat{R}/P\hat{R}$. $Q_0$ corresponds to a minimal prime ideal $\overline{Q}$ of $\hat{R}/P\hat{R} = \hat{\overline{R}}$ with $1 < \dim \hat{\overline{R}}/\overline{Q} = d-1 < n-1 = \dim \overline{R}$. By induction hypothesis $\overline{\Delta}_{i-1} \neq \emptyset$, that is, there is a prime ideal $\overline{P}_0 \subseteq \overline{R}$ with $\mathrm{ht}\,\overline{P}_0 = i-1$ and a prime ideal $\overline{W} \subseteq \hat{\overline{R}}$ which is minimal over $\overline{P}_0\hat{\overline{R}}$ with $\overline{Q} \subseteq \overline{W}$ and $\dim \hat{\overline{R}}/\overline{W} = (d-1)-(i-1) = d-i$. Let $P_0$ and $W$ be the preimages of $\overline{P}_0$ and $\overline{W}$ in $R$ and $\hat{R}$, respectively. Since $R$ is catenary, $\mathrm{ht}\, P = i$ and $W$ is minimal over $P_0\hat{R}$ with $Q \subseteq W$ and $\dim \hat{R}/W = \dim \hat{\overline{R}}/\overline{W} = d-i$. This shows that $\Delta_i \neq \emptyset$.

(14.19) Proposition: Let $(R, m)$ be a local Noetherian domain, $\hat{R}$ its completion, and $(0) = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_r$ a shortest primary decomposition of $(0) \subseteq \hat{R}$ with $\mathcal{J}_i$ $Q_i$-primary for all $1 \le i \le r$. Suppose that ~~let~~ $Q_1 = 0$ and $\dim \hat{R}/Q_1 = 1 < \dim \hat{R}$. Then there are elements $b, c \in m \subseteq R$ and $\hat{d} \in (\mathcal{J}_2 \cap \cdots \cap \mathcal{J}_r) - Q_1$ so that

(a) $b - \hat{d} \in \mathcal{J}_1$

(b) $(b, \hat{d})\hat{R} = (b, c)\hat{R}$

(c) $c/b$ is integral over $R$ and $c/b \notin R$.

Proof: Step 1: By assumption $Q_1$ is a minimal prime ideal of $\hat{R}$ and $r > 1$ since $\dim \hat{R}/Q_1 < \dim \hat{R}$. Moreover, $\mathcal{J}_i \nsubseteq Q_1$ for all $2 \le i \le r$ and there is an element $\hat{a} \in (\mathcal{J}_2 \cap \cdots \cap \mathcal{J}_r) - Q_1$. Since $\dim \hat{R}/Q_1 = 1$, the ideal $\tilde{I} = \mathcal{J}_1 + \hat{a}\hat{R}$ is $m\hat{R}$-primary. In particular, $\tilde{I}$ is extended from $R$, that is, $\tilde{I} = I\hat{R}$ where $I = \tilde{I} \cap R$. Let $b \in I - (0)$ and $\hat{r}, \hat{s} \in \hat{R}$ with $b = \hat{r} + \hat{s}\hat{a}$ where Since $R$ is a domain, $b \notin Q_1$ and therefore $\hat{s}\hat{a} \notin Q_1$. Set $\hat{d} = \hat{s}\hat{a}$, then $b - \hat{d} = \hat{r} \in \mathcal{J}_1 \subseteq Q_1$.

Step 2: Claim: $\hat{d} \notin b\hat{R}$

Pf of Claim: Suppose that $\hat{d} = b\hat{u}$ for some $\hat{u} \in \hat{R}$. Then $b - \hat{d} = b(1 - \hat{u}) \in Q_1$. Since $b \notin Q_1$ and $Q_1$ prime, $1 - \hat{u} \in Q_1 \subseteq m\hat{R}$ and $\hat{u}$ is a unit of $\hat{R}$. This implies that $b \in \mathcal{J}_2 \cap \cdots \cap \mathcal{J}_r$, in particular, $b$ is contained in (another) minimal prime ideal of $\hat{R}$, say $b \in Q_2$ with $Q_2$ minimal in $\hat{R}$. Since $R$ is a domain, $b \in Q_2 \cap R = (0)$ and $b = 0$, a contradiction.

Step 3: Let $\mathcal{J} \subseteq R$ be a nonzero ideal with $m \notin \mathrm{Ass}_R(R/\mathcal{J})$. We claim that $\hat{d} \in \mathcal{J}\hat{R}$. By flatness $(\mathcal{J} : m)\hat{R} = \mathcal{J}\hat{R} : m\hat{R}$ and $\mathcal{J} : m = \mathcal{J}$ since $m \notin \mathrm{Ass}_R(R/\mathcal{J})$. Hence $\mathcal{J}\hat{R} : m\hat{R} = \mathcal{J}\hat{R}$ and $m\hat{R} \notin \mathrm{Ass}_{\hat{R}}(\hat{R}/\mathcal{J}\hat{R})$. Since $Q_1 \subseteq \hat{R}$ is a minimal prime ideal with $\dim \hat{R}/Q_1 = 1$, for all $Q \in \mathrm{Ass}_{\hat{R}}(\hat{R}/\mathcal{J}\hat{R})$: $Q_1 \nsubseteq Q$ and therefore $\mathcal{J}_1 \nsubseteq Q$. Let $\hat{c} \in \mathcal{J}_1 - Q$ and $K$ the $Q$-primary component of $\mathcal{J}\hat{R}$. Then $\hat{c}\hat{d} \in \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_r = (0)$ implying that $\hat{d} \in K$. $\hat{d}$ is in every primary component of $\mathcal{J}\hat{R}$ and therefore $\hat{d} \in \mathcal{J}\hat{R}$.

Step 4: If $m \notin \mathrm{Ass}_R(R/bR)$ then $\hat{d} \in b\hat{R}$ contradicting step 2. Hence

$m \in \mathrm{Ass}_R(R/bR)$ and we can write $bR = I \cap J$ where $I$ is $m$-primary and $J : m = J$. Then $b\hat{R} = I\hat{R} \cap J\hat{R}$ and by step 3 $\hat{d} \in J\hat{R}$. By step 2 $\hat{d} \notin b\hat{R}$ and therefore $\hat{d} \notin I\hat{R}$. Moreover, since $R/I \cong \hat{R}/I\hat{R}$, $I\hat{R}+J\hat{R}/I\hat{R} \cong I+J/I$ and there is an element $c \in J$ with $\hat{d} - c \in I\hat{R}$. Then

$$\hat{d} - c \in I\hat{R} \cap J\hat{R} = b\hat{R}$$

and $(b,c)\hat{R} = (b,\hat{d})\hat{R}$. (b) is proven.

In order to show (c) note that $c \notin b\hat{R}$, since $\hat{d} \notin b\hat{R}$, and therefore $c/b \notin R$. Using $b - \hat{d} \in J_1$ we obtain $\hat{d}(b-\hat{d}) = 0$ and $b\hat{d} = \hat{d}^2$. Moreover, since $\hat{d} - c \in b\hat{R}$, $c = \hat{d} + b\hat{g}$ for some $\hat{g} \in \hat{R}$ and therefore

$$c^2 = \hat{d}^2 + 2b\hat{d}\hat{g} + b^2\hat{g}^2 \in b(b,\hat{d})\hat{R} = b(b,c)\hat{R}.$$

Hence there are $u, v \in R$ so that

$$c^2 - ubc - vb^2 = 0$$

and $c/b$ is integral over $R$.

(14.20) Proposition: Notations and assumptions as in (14.19). Then the ring $S = R[c/b]$ has a maximal ideal of height one.

Proof: Let $T = Q(\hat{R})$ be the total ring of quotients of $\hat{R}$. Since $\hat{R}$ is flat over $R$, $\hat{R} \otimes_R S \subseteq \hat{R} \otimes_R Q(R) \subseteq T$ and since $S$ is a finitely generated $R$-module, $\hat{S} \cong \hat{R} \otimes_R T$ and $\hat{R} \subseteq \hat{S} = \hat{R}[c/b] \subseteq T$ where $\hat{S}$ is the completion of $S$ with respect to the Jacobson radical of $S$. By assumption $\mathrm{Ass}_{\hat{R}}(\hat{R}) = \{Q_1, \ldots, Q_r\}$ with let $Q_1 = 0$ and $\dim \hat{R}/Q_1 = 1$. Set $q_i = Q_iT \cap \hat{S}$, then $\mathrm{Ass}_{\hat{S}}(\hat{S}) = \{q_1, \ldots, q_r\}$ and let $Q_i = $ let $q_i$ since $\hat{S}_{q_i} = T_{Q_iT}$ for all $1 \leq i \leq r$. Moreover, $\hat{S}$ is integral over $\hat{R}$ and $\hat{S}/q_i$ is integral over $\hat{R}/Q_i$ yielding that $\dim \hat{S}/q_i = \dim \hat{R}/Q_i$ for all $1 \leq i \leq r$.

Let $N \subseteq \hat{S}$ be a maximal ideal with $q_1 \subseteq N$. Since $(b,c)\hat{R} = (b,\hat{d})\hat{R}$ we have that $\hat{S} = \hat{R}[c/b] = \hat{R}[\hat{d}/b]$ where $\hat{d} \in Q_2 \cap \ldots \cap Q_r$ and $\hat{d} - b \in Q_1$, $b \notin Q_i$ for all $1 \leq i \leq r$. Thus $\hat{d}/b \in q_2 \cap \ldots \cap q_r$ and $\hat{d}/b - 1 \in q_1$. Thus for all $2 \leq i \leq r$ : $q_1 + q_i = \hat{S}$ and $q_1$ is the only minimal prime ideal contained in $N$. Therefore ht $N = \dim (\hat{S}/q_1)_N \leq \dim \hat{S}/q_1 = \dim \hat{R}/Q_1 = 1$ and ht $N = 1$ since $m\hat{R} \subseteq N$. Then $n = N \cap S$ is a maximal ideal of $S$ with $\widehat{S_n} = \hat{S}_N$ and ht $n = \dim \widehat{S_n} = \dim \hat{S}_N = 1$.

(14.21) <u>Theorem</u>: Let $(R, m)$ be a local Noetherian ring. The following conditions are equivalent:

(a) $R$ is formally catenary.

(b) $R$ is universally catenary.

(c) The polynomial ring $R[x]$ is catenary.

<u>Proof</u>: (a) $\Rightarrow$ (b) : By (14.16)

(b) $\Rightarrow$ (c) : obvious

(c) $\Rightarrow$ (a): Suppose that $R[x]$ is catenary and $R$ is not formally catenary. Then there is a prime ideal $P \subseteq R$ so that $\hat{R}/P\hat{R}$ is not equidimensional. Since $(R/P)[x]$ is also catenary, we replace $R$ by $R/P$ and assume that $R$ is a local Noetherian domain, $\hat{R}$ is not equidimensional, and $R[x]$ is catenary.

Note that $\dim R > 1$ and that there is a minimal prime ideal $Q$ of $R$ with $1 \leq \dim \hat{R}/Q < n = \dim \hat{R}$. If $\dim(\hat{R}/Q) = 1$ set $P_0 = (0) \subseteq R$ and if $\dim(\hat{R}/Q) = d > 1$ by (14.18) there is a prime ideal $P_0 \subseteq R$ so that

($\alpha$) ht $P_0 = d - 1$

($\beta$) there is a minimal prime ideal $W$ of $P_0\hat{R}$ with $\dim(\hat{R}/W) = 1$.

Since $R[x]$ is catenary, $R = R[x]/(x)$ is catenary and by (14.6): $\dim R = \dim R/P_0 + \text{ht } P_0$ and therefore $\dim R/P_0 = n - d + 1 > 1$. Thus we may replace $R$ by $R/P_0$ and assume

($\gamma$) $R$ is a domain with $\dim R > 1$.

($\delta$) $\hat{R}$ has a minimal prime ideal $Q_1$ with $\dim(\hat{R}/Q_1) = 1$.

By (14.19) and (14.20) there is a ring $S \subseteq Q(R)$ with $R \subseteq S$ so that:

(i) $S'$ is integral (finite) over $R$
(ii) $S' = R[\zeta]$ for some $\zeta \in Q(R)$
(iii) $S$ has a maximal ideal $n$ of height one.

Let $f: R[x] \longrightarrow S$ be the surjective $R$-algebra morphism defined by $f|_R = id_R$ and $f(x) = \zeta$. Set $P = \ker(f)$ and $N = f^{-1}(n)$. Since $S'$ is integral over $R$ with $R \subseteq S$, $P \cap R = (0)$ and $n \cap R = m$ is the maximal ideal of $R$.

<u>Claim</u>: $ht\, N = ht\, m + 1$.

<u>Pf of Claim</u>: If $ht\, N \neq ht\, m+1$, then $N = mR[x]$ and $ht\, N = ht\, m$. Then $S/n = (R[x]/N)_N = (R/m)[x]_{(0)} = (R/m)(x)$ and $S/n$ has transcendence degree 1 over $R/m$. This contradicts that $S'$ is integral over $R$ and hence $S/n$ algebraic over $R/m$.

Since $\dim R[x] = \dim R + 1 > \dim S'$, it follows that $P \neq (0)$ and, since $R[x]$ is catenary, $\dim R[x] = \dim R + 1 = \dim R[x]/P + ht\, P = \dim S' + ht\, P$ and $ht\, P = 1$. Moreover, by (14.8)

$$ht\,(N/P) = ht\, n = 1 = ht\, N - ht\, P = (ht\, m + 1) - 1$$
$$= ht\, m = \dim R > 1,$$

a contradiction. Thus $R$ is formally catenary.

(14.22) <u>Corollary</u>: A Noetherian ring $R$ is universally catenary if and only if the polynomial ring $R[x]$ is catenary.

<u>Proof</u>: "$\Leftarrow$": $R$ is universally catenary if and only if $R_P$ is universally catenary for all prime ideals $P \subseteq R$. Let $P \in Spec(R)$ then by assumption $R[x]$ is catenary and hence $R_P[x]$ is catenary. By (14.21) $R_P$ is universally catenary.

(14.23) <u>Corollary</u>: Let $R$ be a Noetherian ring.
(a) If $\dim R \leq 1$, then $R$ is universally catenary.
(b) If $\dim R = 2$, then $R$ is catenary.

<u>Proof</u>: (a) follows by (b) and (14.22) since $\dim R[x] = \dim R + 1 \leq 2$.

(14.24) <u>Theorem</u>: Let $(R, m)$ be a local Noetherian universally catenary ring and let $(S, n)$ be a local Noetherian subring of the $m$-adic completion $\widehat{R}$ with $R \subseteq S \subseteq \widehat{R}$ and $\widehat{S} = \widehat{R}$ where $\widehat{S}$ is the $n$-adic completion of $S$. Then $S$ is universally catenary.

<u>Proof</u>: By (14.21) it suffices to show that $S$ is formally catenary, i.e. for all $P \in \operatorname{Spec}(S)$ the ring $\widehat{R}/P\widehat{R}$ is equidimensional. We may assume that $P \cap R = (0)$ and hence that $R$ is a domain. Let $Q$ and $W$ be minimal primes over $P\widehat{R}$. We have to show that $\dim(\widehat{R}/Q) = \dim(\widehat{R}/W)$.

Since $S$ is Noetherian, the natural maps $S_P \longrightarrow \widehat{R}_Q$ and $S_P \longrightarrow \widehat{R}_W$ are flat. By 911, (8.63):

$\dim(\widehat{R}_Q) = \dim S_P + \dim(\widehat{R}_Q/P\widehat{R}_Q)$ and $\dim(\widehat{R}_W) = \dim S_P + \dim(\widehat{R}_W/P\widehat{R}_W)$.

Since $Q$ and $W$ are minimal over $P\widehat{R}$ it follows that

$$\dim(\widehat{R}_Q) = \dim(\widehat{R}_W) = \dim S_P$$

Let $Q_0 \subseteq Q$ and $W_0 \subseteq W$ be minimal prime ideals of $\widehat{R}$ so that $\dim(\widehat{R}_Q) = \dim(\widehat{R}_Q/Q_0\widehat{R}_Q)$ and $\dim(\widehat{R}_W) = \dim(\widehat{R}_W/W_0\widehat{R}_W)$. Since $R$ is a universally catenary domain, its completion $\widehat{R}$ is equidimensional and therefore

$$\dim(\widehat{R}/Q_0) = \dim(\widehat{R}/W_0).$$

Since $\widehat{R}$ is catenary, by (14.6):

$\dim(\widehat{R}/Q_0) = \operatorname{ht}(Q/Q_0) + \dim(\widehat{R}/Q)$ and $\dim(\widehat{R}/W_0) = \operatorname{ht}(W/W_0) + \dim(\widehat{R}/W)$.

Since $\operatorname{ht}(Q/Q_0) = \dim(\widehat{R}_Q/Q_0\widehat{R}_Q) = \dim(\widehat{R}_Q)$, $\operatorname{ht}(W/W_0) = \dim(\widehat{R}_W/W_0\widehat{R}_W) = \dim(\widehat{R}_W)$, and $\dim(\widehat{R}_Q) = \dim(\widehat{R}_W)$, it follows that $\dim(\widehat{R}/Q) = \dim(\widehat{R}/W)$.

(14.25) <u>Corollary</u>: Let $(R, m)$ be a local Noetherian universally catenary ring. Then the Henselization $R^h$ of $R$ is universally catenary.

<u>A note on examples:</u>

The following theorem is mentioned without proof:

(14.26) <u>Theorem</u>: Let $K$ be a countable field of infinite transcendence degree over a prime field and let $R = K[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)}$ be the localized polynomial ring in $n$ variables. Let $m \in \mathbb{N}$ be an integer with $m < n$ and $I \subseteq R$ an ideal which satisfies the following condition:

(*) For all associated prime ideals $Q \in \mathrm{Ass}(\hat{R}/I\hat{R})$ it holds that $Q \subseteq (z_1, \ldots, z_m)\hat{R}$ where $m < n$.

Then there is a local Noetherian domain $S$ with $R \subseteq S \subseteq Q(R)$ and the following properties:

(a) There is a $K$-algebra isomorphism $\varphi : \hat{S} \longrightarrow \hat{R}/I\hat{R}$.

(b) For every nonzero prime ideal $P \subseteq S$ the ring $S/P$ is essentially of finite type over $K$.

(c) The prime ideal $\hat{P}_0 = \varphi^{-1}(z_1, \ldots, z_m) \subseteq \hat{S}$ lies over $(0)$, that is, $\hat{P}_0 \cap S = (0)$.

Using this theorem one can construct a variety of local Noetherian domains $S$ which fail to be formally equidimensional. Actually, in 1982 T. Ogoma provided a counterexample to Nagata's chain conjecture by using a similar construction as in the theorem. Nagata's chain conjecture states that the normalization of a Noetherian domain is universally catenary. Ogoma constructs a local Nagata domain $S$ whose completion $\hat{S}$ fails to be equidimensional. More precisely, let $R = K[x,y,z,u]_{(x,y,z,u)}$ and $I = (xy, xz) \subseteq R$. The Noetherian domain $S$ from Theorem (14.26) with completion $\hat{S} \cong K[[x,y,z,u]]/(xy,xz)$ is Ogoma's example. Theorem (14.26) can also be used to construct a local Nagata domain with a non-open regular locus.