CHAPTER XVII: EXCELLENT RINGS AND ARTIN APPROXIMATION

Recall the following definition:

(17.1) Definition: Let R be a semilocal Noetherian ring and $\widehat{R}$ its completion with respect to the Jacobson radical. R has the approximation property if every system of equations over R which has a solution in $\widehat{R}$ is solvable in R. More precisely, let $X = (x_1, -, x_n)$ be variables and $(f) = (f_1, -, f_N) \subseteq R[X]$ polynomials over R. If there is an n-tuple $\hat{y} = (\hat{y}_1, -, \hat{y}_n) \in \widehat{R}^n$ with $f_i(\hat{y}) = 0$ for all $1 \le i \le N$, then there is an n-tuple $y = (y_1, -, y_n) \in R^n$ with $f_i(y) = 0$ for all $1 \le i \le N$.

(17.2) Remark: (a) A local Noetherian ring with approximation property is Henselian.
(b) Let (R, m) be a local Noetherian ring with approximation property, $(f) = (f_1, -, f_N) \in R[X]$, and $\hat{y} = (\hat{y}_1, -, \hat{y}_n) \in \widehat{R}^n$ with $f_i(\hat{y}) = 0$ for all $1 \le i \le N$. Then for all $t \in \mathbb{N}$ there is an n-tuple $y_t = (y_{t1}, -, y_{tn}) \in R^n$ with $f_i(y_t) = 0$ and $y_{tj} \equiv \hat{y}_j \mod m^t \widehat{R}$ for all $1 \le i \le N$, $1 \le j \le n$.

A semilocal Noetherian ring R has the approximation property if and only if R is excellent and Henselian. In this chapter we want to prove the forward direction. The proof of the backward direction is much harder. It requires the solution of the following conjecture:

Artin conjecture: Let $\varphi: R \longrightarrow S$ be a regular morphism of Noetherian rings. Then S is a direct limit of smooth R-algebras of finite type.

The following theorem is a special case of Artin's conjecture:

(17.3) Theorem: Let R be an excellent discrete valuation ring. Then the formal power series ring $R[\![x_1, \ldots, x_n]\!]$ is a direct limit of smooth R-algebras of finite type.

The proof of (17.3) is easier than the proof of the full Artin conjecture. It uses Jacobian criteria which are available for rings of finite type over a discrete valuation ring. We will use Theorem (17.3) without proof in the remainder of this section.

§1: PRELIMINARIES

(17.4) Proposition: Let R be a semilocal ring with approximation property. If R is reduced (a domain), then $\hat{R}$ is reduced (a domain).

Proof: Apply the approximation property to the equation $x^t = 0$ in the reduced case and to $xy = 0$ in the domain case.

(17.5) Proposition: Let R be a semilocal ring with approximation property and S a finite R-algebra. Then S has the approximation property.

Proof: Suppose that $S = \sum_{i=1}^{r} R w_i$ and consider the surjective R-linear map

$$(*) \qquad R^r \xrightarrow{\varphi} S \longrightarrow 0$$

defined by $\varphi(e_i) = w_i$ where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^r$.

Suppose that $\ker \varphi$ is generated by $z_1, \ldots, z_s$ with $z_j = \begin{bmatrix} z_{j1} \\ \vdots \\ z_{jr} \end{bmatrix} \in R^r$. Tensoring $(*)$ with $\hat{R}$ yields an exact sequence:

$$\hat{R}^r \xrightarrow{\hat{\varphi}} \hat{S} \longrightarrow 0$$

with $\ker \hat{\varphi} = (\ker \varphi)\hat{R} = \sum_{i=1}^{s} \hat{R} z_i$.

Let $(F_i)_{i=1,\dots,t} \subseteq S[y_1,\dots,y_N]$ be polynomials and $\hat{s} = (\hat{s}_1,\dots,\hat{s}_N) \in \hat{S}^N$ with $F_i(\hat{s}) = 0$ for all $1 \le i \le t$. Introduce new variables y_{ij}, $1 \le i \le N$, $1 \le j \le r$ and set

$$y_i = y_{i1} w_1 + \dots + y_{ir} w_r .$$

Then write every element $b \in S$ in the form $b = \sum_{i=1}^{r} a_i w_i$ with $a_i \in R$ and substitute:

$$F_i(y_1,\dots,y_N) = \sum_{j=1}^{r} f_{ij}(y_{11},\dots,y_{Nr})\, w_j$$

with polynomials $f_{ij} \in R[y_{11},\dots,y_{Nr}]$. By assumption there is an N-tuple $\hat{s} = (\hat{s}_1,\dots,\hat{s}_N) \in \hat{S}^N$ with $F_i(\hat{s}) = 0$ for all $1 \le i \le t$. Write

$$\hat{s}_i = \sum_{j=1}^{r} \hat{s}_{ij}\, w_j$$

where $\hat{s}_{ij} \in \hat{R}$. This yields

$$0 = F_i(\hat{s}) = \sum_{j=1}^{r} f_{ij}(\hat{s}_{11},\dots,\hat{s}_{Nr})\, w_j$$

and for all $1 \le i \le t$ the r column

$$\begin{bmatrix} f_{i1}(\hat{s}_{11},\dots,\hat{s}_{Nr}) \\ \vdots \\ f_{ir}(\hat{s}_{11},\dots,\hat{s}_{Nr}) \end{bmatrix} \in \ker \hat{\varphi} = \sum_{j=1}^{r} \hat{R} z_j = \sum_{j=1}^{r} \hat{R} \begin{bmatrix} z_{j1} \\ \vdots \\ z_{jr} \end{bmatrix}.$$

Thus we may write

$$\begin{bmatrix} f_{i1}(\hat{s}_{11},\dots,\hat{s}_{Nr}) \\ \vdots \\ f_{ir}(\hat{s}_{11},\dots,\hat{s}_{Nr}) \end{bmatrix} = \sum_{j=1}^{s} \begin{bmatrix} z_{j1} \\ \vdots \\ z_{jr} \end{bmatrix} \hat{\ell}_{ij} = \begin{bmatrix} \sum_{j=1}^{s} \hat{\ell}_{ij} z_{j1} \\ \vdots \\ \sum_{j=1}^{s} \hat{\ell}_{ij} z_{jr} \end{bmatrix}.$$

Hence the system of equations

$$(**) \qquad G_{ik}(y_{11},\dots,y_{Nr}, x_{11},\dots,x_{ts}) = f_{ik}(y_{11},\dots,y_{Nr}) - \sum_{j=1}^{s} z_{jk} x_{ik} = 0$$

in the variables $y_{11},\dots,y_{Nr}, x_{11},\dots,x_{ts}$ has a solution in $\hat{R}$. Since R has the approximation property, $(**)$ is solvable in R which yields a solution of $F_i = 0$ in $\hat{S}$.

(17.6) <u>Definition</u>: Let R be a semilocal Noetherian ring and $\hat{R}$ its completion with respect to the Jacobson radical. R has <u>geometrically</u> <u>normal</u> <u>formal</u> <u>fibers</u> if for all $P \in \operatorname{Spec}(R)$ and all finite field extensions $k(P) \subseteq L$ the ring $\hat{R} \otimes_R L$ is normal.

(17.7) <u>Remark</u>: Let R be a semilocal Noetherian ring, $\hat{R}$ its completion with respect to the Jacobson radical.

(a) If R is normal with geometrically normal formal fibers then $\hat{R}$ is normal.

(b) Suppose that R has geometrically normal formal fibers. Then every semilocal R-algebra essentially of finite type over R has geometrically normal formal fibers.

(c) The following conditions are equivalent:

(i) R has geometrically reduced (normal) formal fibers

(ii) For every finite R-algebra S which is a domain and all prime ideals $\hat{Q} \subseteq \hat{S}$ with $\hat{Q} \cap S = 0$ the ring $\hat{S}_{\hat{Q}}$ is reduced (normal).

(d) If R has geometrically reduced formal fibers the following are equivalent:

(i) R has geometrically normal formal fibers.

(ii) For every finite R-algebra S which is a normal domain its completion $\hat{S}$ is normal.

— without proof

(17.8) <u>Corollary</u>: Let R be a local ring with approximation property. Then R has geometrically reduced formal fibers, or equivalently, R is a Nagata ring.

<u>Proof</u>: If S is a finite R-algebra and a domain, then S has the approximation property by (17.5). Thus $\hat{S}$ is a domain by (17.4). By (17.7)(c) R has geometrically reduced formal fibers.

(17.9) <u>Corollary</u>: Let (R,m) be a local ring with approximation property. Then R has geometrically normal formal fibers.

<u>Proof</u>: Let S be a finite R-algebra which is a normal domain. Since R is

Henselian, $\hat{S}$ is a local ring. By (17.5) S has the approximation property and thus $\hat{S}$ is a domain by (17.4). By (17.7)(d) we have to show that $\hat{S}$ is normal. Let $\hat{a}, \hat{b} \in \hat{S}$, $\hat{b} \neq 0$, with $\hat{a}/\hat{b}$ integral over $\hat{S}$, that is, $\hat{a}/\hat{b}$ satisfies an integral equation:

$$(\hat{a}/\hat{b})^n + \hat{c}_{n-1}(\hat{a}/\hat{b})^{n-1} + \dots + \hat{c}_0 = 0$$

where $\hat{c}_i \in \hat{S}$. This yields

$$\hat{a}^n + \hat{c}_{n-1}\hat{a}^{n-1}\hat{b} + \dots + \hat{c}_0 \hat{b}^n = 0.$$

Consider the following equation over R:

$$x^n + z_{n-1}x^{n-1}y + \dots + z_0 y^n = 0.$$

Since R has the approximation property, for all $k \in \mathbb{N}$ there are elements $a_k, b_k, c_{ik} \in R$ with

$$a_k^n + c_{n-1,k}\, a_k^{n-1} b_k + \dots + c_{0k} b_k^n = 0$$

and $a_k \equiv \hat{a} \bmod n^k \hat{S}$, $b_k \equiv \hat{b} \bmod n^k \hat{S}$ where $n \subseteq S$ is the maximal ideal. Since S is normal, for all $k \in \mathbb{N}$ $a_k \in S b_k$ and thus $\hat{a} \in \hat{S}\hat{b} + n^k \hat{S}$. This implies that $\hat{a} \in \hat{S}\hat{b}$ and $\hat{S}$ is normal.

(17.10) Corollary: Let R be a local Noetherian ring with approximation property. Then R is universally catenary.

Proof: The proof of (15.35) only uses that the formal fibers of $R = R^h$ are geometrically normal.

§2: THE EQUATIONS

We want to show:

(17.11) Theorem: Let (R, m) be a local Noetherian ring with approximation property. Then R is excellent.

By section 1 we only have to show that R is a G-ring. By (15.19) R is a G-ring if and only if every finite R-algebra S which is a domain satisfies the following condition: for all $Q \in \mathrm{Sing}(\hat{S})$ the intersection $Q \cap S \neq 0$ is not trivial. Since R is Henselian, S is a local domain and by (17.7) S has the approximation property. Thus it suffices to show:

(*) Let R be a local domain with approximation property and $Q \subseteq \hat{R}$ a prime ideal with $\hat{R}_Q$ not regular. Then $Q \cap R \neq 0$.

It is very hard (maybe impossible) to express the singularity of $\hat{R}$ at Q in terms of equations over R. Therefore we first study the following situation:

(17.12) Let $\psi: T_0 \longrightarrow T$ be a faithfully flat morphism of regular local rings. Suppose that the maximal ideal m_T of T is generated by the maximal ideal m_0 of T_0. Fix prime ideals Q and P of T with $Q \subseteq P$ and $(T/Q)_P$ not regular.

Roughly speaking the aim of this section is to construct a T_0-algebra B of finite type together with finitely many elements $\{Q_\lambda\}_{\lambda\in\Lambda}$ and $\{Q_\sigma\}_{\sigma\in\Sigma}$ (where $\Lambda \subseteq \Sigma$) of B such that the following holds:

For 'certain' T_0-algebra morphisms $\varphi: B \longrightarrow T$ the following conditions are satisfied:

(a) The elements $\varphi(Q_\lambda)$, $\lambda \in \Lambda$, generate the T-ideal Q.

(b) If $\tilde{P} \subseteq T$ is a minimal prime ideal over $(\varphi(Q_\sigma))_{\sigma\in\Sigma}$, then the ring $(T/\mathfrak{a})_{\tilde{\mathfrak{p}}}$ is a non-regular domain.

(17.13) Remark and notation: In the following we study relations in T which will be transformed into equations over T_0. By writing

(*) $$\underline{q}_\nu = \sum_{\mu \neq \nu} \underline{a}_{\nu\mu}\, \underline{q}_\mu + \sum_\lambda m_\lambda \underline{c}_{\nu\lambda}$$

we indicate that (*) should be read in the following two different ways:

(a) (*) is a relation in T with underlined elements $q_\nu, a_{\nu\mu}, q_\mu, c_{\nu\lambda}$ in T and not underlined elements m_λ in T_0.

(b) (*) is an equation over T_0 where the underlined elements $\underline{q}_\nu, \underline{a}_{\nu\mu}, \underline{q}_\mu, \underline{c}_{\nu\lambda}$ are variables over T_0 (in the ongoing text replaced by capital letters) and the not underlined elements m_λ of T_0 are unchanged.

We also consider the polynomial ring $T_0[A_{\nu\mu}, C_{\nu\mu}, Q_\nu]$ modulo the ideal I generated by the elements $Q_\nu - \sum_{\mu\neq\nu} A_{\nu\mu} Q_\mu - \sum_\lambda m_\lambda C_\lambda$. The ring $T_0[A_{\nu\mu}, C_{\nu\mu}, Q_\nu]/I$ is called the T_0-algebra corresponding to the system of equations (*).

We refer to (*) as to relations in T or as to equations over T_0 to indicate in which way (*) should be read. Obviously the equations (*) over T_0 have solutions $q_\nu, a_{\nu\mu}, q_\mu, c_{\nu\mu}$ in T.

The following lemma is frequently used:

(17.14) Lemma: Let $\varphi: T_0 \longrightarrow T$ be as above with $\dim T_0 = \dim T = n$. Let $I \subseteq T$ be an ideal with ht $I = k$. Then there are $n-k$ elements $\ell_1, \ldots, \ell_{n-k} \in m_0 \subseteq T_0$ so that the T-ideal $I + (\varphi(\ell_1), \ldots, \varphi(\ell_{n-k}))T$ is m_T-primary, i.e. $\varphi(\ell_1), \ldots, \varphi(\ell_{n-k})$ form a system of parameters of T/I.

Proof: Immediately by induction on $n-k$.

Equations describing Q

(17.15) For every nonzero element $a \in T$ there is a $k \in \mathbb{N}$ with $a \in m_T^k - m_T^{k+1}$. This integer k is called the degree of a and is denoted by $\nu(a) = k$. Consider the map

$$\Theta : T-(0) \longrightarrow gr_{m_T}(T)$$

defined by $\Theta(a) = a^* = a + m_T^{k+1} \in m_T^k/m_T^{k+1}$ where $k = \nu(a)$. The element a^* is called the leading form of a. If $I \subseteq T$ is an ideal then $I^* \subseteq gr_{m_T}(T)$ denotes the ideal of $gr_{m_T}(T)$ which is generated by the leading forms of the elements of I. Note that if $I = (a_1, —, a_t)$ then in general $I^* \neq (a_1^*, —, a_t^*)$.

(17.16) Choose a regular system of parameters $x_1, —, x_n \in m_o \subseteq T_o$. Notice that $x_1, —, x_n$ (more precisely: $\psi(x_1), —, \psi(x_n)$) forms a regular system of parameters of T. For all $k \in \mathbb{N}$ consider the following subsets of T_o:

$$\Gamma_k = \{x_1^{e_1} \dots x_n^{e_n} \mid \textstyle\sum e_i = k\} = \{m(k,\mu) \mid \mu = 1, —, r_k\}$$

where $m(k,\mu)$ is a monomial of degree k and μ denotes a enumeration of Γ_k. Let $k_o = T_o/m_o$ and $k = T/m$ denote the residue class fields of T_o and T. Since T_o and T are regular the sets $\{m(k,\mu) + m_o^{k+1}\}_{1 \le \mu \le r_k}$ and $\{m(k,\mu) + m_T^{k+1}\}_{1 \le \mu \le r_k}$ form a basis of the k_o (resp. k)-vector space m_o^k/m_o^{k+1} or m_T^k/m_T^{k+1}.

(17.17) Let $\{q_\lambda\}_{\lambda \in \Lambda}$ be a finite subset of Q so that:
(a) $\{q_\lambda\}_{\lambda \in \Lambda}$ generate the T-ideal Q
(b) $\{q_\lambda^*\}_{\lambda \in \Lambda}$ generate the $gr_{m_T}(T)$-ideal Q^*.

(17.18) Let $t := \max\{\nu(q_\lambda) \mid \lambda \in \Lambda\}$. For all $k \le t$ let $W(k) \subseteq m_T^k/m_T^{k+1}$ denote the k-subspace of m_T^k/m_T^{k+1} which is generated by all elements q_λ^*

with $\nu(q_\lambda) = k$ and $q_\lambda^* m(\gamma,\mu)$ where $\nu(q_\lambda) + \gamma = k$. Thus for all $k \le t$ there is a finite set of elements of T:

$$\Pi_k = \{ q_\lambda \mid \nu(q_\lambda) = k \} \cup \{ q_\lambda m(\gamma,\mu) \mid \nu(q_\lambda) + \gamma = k \}$$

so that the set of leading forms

$$\Pi_k^* = \{ q_\lambda^* \} \cup \{ q_\lambda^* m(\gamma,\mu) \}$$

forms a basis of the k-subspace $W(k)$. For all $k \le t$ choose a finite subset $\Theta_k \subseteq \Gamma_k$ so that $\Pi_k^* \cap \Theta_k = \emptyset$ and $\Pi_k^* \cup \Theta_k$ is a basis of m_T^k / m_T^{k+1}. In particular, $|\Pi_k| + |\Theta_k| = \dim (m_T^k / m_T^{k+1})$.

(17.19) The first set of relations/equations describes the degrees of the generating elements q_λ. To do so consider for all $\lambda \in \Lambda$ relations

(17.19)(a) $$\underline{q_\lambda} = \sum_{m(\nu(q_\lambda),\mu) \in \Gamma_{\nu(q_\lambda)}} \underline{a}_{\lambda\mu}\, m(\nu(q_\lambda),\mu)$$

where $a_{\lambda\mu} \in T$. The next set of equations describes the fact that $\Pi_k^* \cup \Theta_k$ forms a basis of Γ_k. For all $m(k,\mu) \in \Gamma_k - \Theta_k$ there are relations

(17.19)(b) $$m(k,\mu) = \sum_{q_\alpha \in \Pi_k} \underline{b}_{(k,\mu)\alpha}\, \underline{q_\alpha} + \sum_{q_\beta m(\gamma,\sigma_\beta) \in \Pi_k} \underline{c}_{(k,\mu)\beta}\, q_\beta\, m(\gamma,\sigma_\beta)$$
$$+ \sum_{m(k,\rho) \in \Theta_k} \underline{d}_{(k,\mu)\gamma}\, m(k,\rho_\gamma) +$$
$$+ \sum_{m(k+1,\sigma) \in \Gamma_{k+1}} \underline{e}_{(k+1,\mu)\delta}\, m(k+1,\delta).$$

By changing the underlined elements into variables we obtain a system of equations over T_0. The variables replacing $\{q_\lambda\}_{\lambda\in\Lambda}$ are denoted by $\{Q_\lambda\}_{\lambda\in\Lambda}$.

Let B_1 be the T_0-algebra of finite type which corresponds to the relations (17.19)(a) and (17.19)(b). B_1 has the following property:

(17.20) <u>Proposition</u>: Let $\varphi : B_1 \longrightarrow T$ be a T_0-algebra morphism with $\varphi(Q_\lambda) \subseteq Q$ for all $\lambda \in \Lambda$. Then Q is generated (as a T-ideal) by $\{\varphi(Q_\lambda)\}_{\lambda\in\Lambda}$.

Proof: Set $\mathcal{J} = (\varphi(Q_\lambda))_{\lambda\in\Lambda} \subseteq Q$. Then $\mathcal{J}^* \subseteq Q^*$ and by (17.19)(a) for all $\lambda \in \Lambda$ $\nu(\varphi(Q_\lambda)) \geq \nu(q_\lambda)$. If $k \leq t$, then the leading forms of $\mathcal{J}^*$ of degree k are contained in $W(k) = Q^* \cap m_T^k/m_T^{k+1}$ and by (17.19)(b) they generate $W(k)$. Hence $\mathcal{J} \subseteq Q$ with $\mathcal{J}^* = Q^*$ which implies that $\mathcal{J} = Q$.

Equations describing ht P in T

(17.21) Since $Q \subseteq P$, we enlarge the system of generators $\{q_\lambda\}_{\lambda\in\Lambda}$ of Q to a finite system of generators $\Omega = \{q_\sigma\}_{\sigma\in\Sigma}$ of P where $\Lambda \subseteq \Sigma$. By further enlarging Ω (if necessary) we may assume that there are elements $v_1, -, v_\ell \in \Omega = \{q_\sigma\}_{\sigma\in\Sigma}$ so that

(a) $sP \subseteq (v_1, —, v_\ell)T$ for some $s \in T - P$

(b) $ht(v_1, —, v_\ell)T = \ell$ (Note : T is regular).

Since T is a regular local ring of dimension n, condition (b) is equivalent to the existence of $n-\ell$ elements $s_1, —, s_{n-\ell} \in T$ so that the ideal $(v_1, —, v_\ell, s_1, —, s_{n-\ell})$ is m_T-primary. By Lemma (17.14) we may choose $s_1, —, s_{n-\ell} \in T_0$. Thus there is a $k \in \mathbb{N}$ so that

$$m_T^k = (x_1, —, x_n)^k T \subseteq (v_1, —, v_\ell, s_1, —, s_{n-\ell})T.$$

This translates into relations:

(17.21)(a) $\underline{v_i} = q_{\sigma_i}$ for all $1 \leq i \leq \ell$ and some $\sigma_i \in \Sigma$

(17.21)(b) $\underline{s}\, q_\sigma = \sum_{i=1}^{\ell} \underline{f}_{\sigma i}\, \underline{v_i}$ for all $\sigma \in \Sigma$

(17.21)(c) $m(k,\mu) = \sum_{i=1}^{\ell} \underline{g}_{\mu i}\, \underline{v_i} + \sum_{j=1}^{n-\ell} \underline{h}_{\mu j}\, s_j$ for all $m(k,\mu) \in \Gamma_k$.

Replace the underlined elements of (17.21)(a),(b),(c) by capital letters and let B_2 be the T_0-algebra corresponding to relations (17.19)(a),(b) and (17.21)(a),(b),(c).

(17.22) Proposition: Let $\varphi: B_2 \longrightarrow T$ be a T_0-algebra morphism with the property that $\varphi(S) \notin (\varphi(V_i))_{1\leq i\leq \ell}\, T$. Then the ideal $(\varphi(Q_\sigma)_{\sigma\in\Sigma})T$ has a minimal prime divisor P_φ with $ht\, P_\varphi = ht\, P = \ell$.

Proof: By (17.21)(c): $m_T^k \subseteq (\varphi(V_1), \ldots, \varphi(V_\ell), s_1, \ldots, s_{n-\ell})T$ and $\varphi(V_1), \ldots, \varphi(V_\ell)$ is part of a regular system of parameters of T. In particular, $\operatorname{ht}(\varphi(V_i))_{1\le i\le \ell}T = \ell$ and the ideal $(\varphi(V_i))_{1\le i\le \ell}T$ is unmixed.

Set $I = (\varphi(Q_\sigma))_{\sigma\in\Sigma}T$. Since $\varphi(S) \notin (\varphi(V_i))_{1\le i\le \ell}T$ there is a primary component $\mathfrak{r}$ of $(\varphi(V_i))_{1\le i\le \ell}T$ with $\varphi(S)\notin \mathfrak{r}$. By (17.21)(b) $\varphi(S)I \subseteq \mathfrak{r}$ and hence $I^t \subseteq \mathfrak{r}$ for some $t\in\mathbb{N}$. Suppose that $\mathfrak{r}$ is P_φ-primary. Then $I\subseteq P_\varphi$ and $\operatorname{ht} P_\varphi = \operatorname{ht}(\varphi(V_i))_{1\le i\le \ell}T = \ell$, since $(\varphi(V_i))_{1\le i\le \ell}T$ is unmixed. Thus P_φ is minimal over $(\varphi(Q_\sigma)_{\sigma\in\Sigma})T = I$.

<u>Equations describing the singularity of $(T/Q)_P$</u>

(17.23) Suppose that $|\Lambda| = m$ with $\Lambda = \{1, \ldots, m\}$ and $\{q_\lambda\}_{\lambda\in\Lambda} = \{q_1, \ldots, q_m\}$. Let S_m denote the symmetric group on m letters. Then $(T/Q)_P$ is regular if and only if there is a $\pi\in S_m$ with $q_{\pi(1)}, \ldots, q_{\pi(r)}$ where $r = \operatorname{ht} Q$ is part of a regular system of parameters of T_P. This is equivalent to the following condition: For all $t\in T-P$ and all $1\le i<r$:

$$t\, q_{\pi(i+1)} \notin P^2 + (q_{\pi(1)}, \ldots, q_{\pi(i)}) \quad \text{for some } \pi\in S_m.$$

By assumption the ring $(T/Q)_P$ is singular. Thus for all $\pi\in S_m$ there is an integer $k(\pi)\in\mathbb{N}\cup\{0\}$ with $k(\pi) < r = \operatorname{ht} Q$ and an element $z(\pi)\in T-P$ so that

$$z(\pi)\, q_{\pi(k(\pi)+1)} \in P^2 + (q_{\pi(1)}, \ldots, q_{\pi(k(\pi))}).$$

With $z = \prod_{\pi\in S_m} z(\pi)$ we obtain for all $\pi\in S_m$ relations in T:

(17.23)(a) $$z\, q_{\pi(k(\pi)+1)} = \sum_{\sigma,\tau\in\Sigma} f_{\pi\sigma\tau}\, q_\sigma q_\tau + \sum_{\mu=1}^{k(\pi)} \ell_{\pi\mu}\, q_{\pi(\mu)}.$$

The fact that $z\notin P$ can be described by $\operatorname{ht}(P,z) = \ell+1$ and $\operatorname{ht} P = \ell$. Thus by Lemma (17.14) there are $n-\ell-1$ elements $u_1, \ldots, u_{n-\ell-1} \in m_0 \subseteq T_0$ so that

$$m_T^s \subseteq P + Tz + (u_1, \ldots, u_{n-\ell-1})T$$

for some $s \in \mathbb{N}$. This yields for all $m(s,\varrho) \in \Gamma_s$ relations in T:

(17.23)(b) $$m(s,\varrho) = \sum_{\sigma \in \Sigma} \underline{o}_{\varrho\sigma}\, q_\sigma + \underline{r}_\varrho\, \underline{z} + \sum_{i=1}^{n-\ell-1} g_{\varrho i}\, u_i .$$

We transform relations (17.19)(a),(b); (17.21)(a),(b),(c) and (17.23)(a),(b) into equations over T_0. As usual we suppose that elements $\{q_\sigma\}_{\sigma\in\Sigma}$, $\{v_i\}_{1\le i\le \ell}$, s, z are transformed into variables $\{Q_\sigma\}_{\sigma\in\Sigma}$, $\{V_i\}_{1\le i\le \ell}$, S, Z. Let B_3 be the T_0-algebra of finite type corresponding to this system of equations.

(17.24) Proposition: Let $\varphi: B_3 \longrightarrow T$ be a T_0-algebra morphism satisfying:
(i) $\varphi(Q_\lambda) \in Q$ for all $\lambda \in \Lambda$
(ii) $\varphi(S) \notin (\varphi(V_1), \ldots, \varphi(V_\ell))T$.
Then the T-ideal generated by $\{\varphi(Q_\sigma)\}_{\sigma\in\Sigma}$ has a minimal prime divisor P_φ with properties:
(a) $Q \subseteq P_\varphi$
(b) $ht\, P_\varphi = \ell$
(c) the local ring $(T/Q)_{P_\varphi}$ is singular.

Proof: Set $I = (\varphi(Q_\sigma))_{\sigma\in\Sigma}$. Because of assumption (i) by (17.20) the ideal generated by $\{\varphi(Q_\lambda)\}_{\lambda\in\Lambda}$ is exactly Q, hence $Q \subseteq I$. Moreover, (ii) implies by (17.22) that there is a minimal prime divisor P_φ of I with $ht\, P_\varphi = \ell$.

It remains to show the $(T/Q)_{P_\varphi}$ is singular. First note that by (17.23)(b)
$$m_T^s \subseteq (P_\varphi, \varphi(Z)) + (u_1, \ldots, u_{n-\ell-1})T$$
and therefore $ht\,(P_\varphi, \varphi(Z)) \ge \ell + 1$, in particular $\varphi(Z) \notin P_\varphi$.

Obviously, the ring $(T/Q)_{P_\varphi}$ is regular if and only if the set of generators $\{\varphi(Q_1), \ldots, \varphi(Q_m)\}$ of Q contains part of a regular system of parameters of the regular ring T_{P_φ}. Since $ht\, Q = r$, the ring $(T/Q)_{P_\varphi}$ is regular if and only if there is a $\pi \in S_m$ so that $\varphi(Q_{\pi(1)}), \ldots, \varphi(Q_{\pi(r)})$ is part of a regular system of parameters of T_{P_φ}. By (17.23)(a) for all $\pi \in S_m$

there is an integer $k(\pi) < r$ so that

$$\varphi(Z)\,\varphi(Q_{\pi(k(\pi)+1)}) \in P_\varphi^2 + (\varphi(Q_{\pi(1)}), \ldots, \varphi(Q_{\pi(k)}))T.$$

Since $\varphi(Z) \notin P_\varphi$, the local ring $(T/Q)_{P_\varphi}$ is singular.

Relations with other minimal prime ideals of Sing (T/Q)

(17.25) From now on we assume that the singular locus of T/Q is closed in $\operatorname{Spec}(T/Q)$ and that P is a minimal prime ideal of Sing (T/Q). If P has minimal height among the (finitely many) minimal prime ideals of Sing (T/Q) no further equations are necessary. Otherwise let $\{W_1, \ldots, W_r\} \subseteq \operatorname{Spec}(T)$ be the preimages of the minimal prime ideals of Sing (T/Q) with $ht\, W_i < ht\, P = l$ for all $1 \le i \le r$ and pick an element $c \in \prod_{i=1}^{r} W_i - P$. Since $c \notin P$, $ht(P,c) = l+1$ and there are elements $t_1, \ldots, t_{n-l-1} \in m_0 \subseteq T_0$ such that

$$m_T^t \subseteq (P, c, t_1, \ldots, t_{n-l-1})T.$$

This translates into relations for all $m(t,\mu) \in \Gamma_t$:

$$(17.25)(a) \qquad m(t,\mu) = \sum_{\sigma\in\Sigma} \underline{v}_{\mu\sigma}\, \underline{q}_\sigma + \underline{w}_\mu\, \underline{c} + \sum_{i=1}^{n-t-1} \underline{z}_{\mu i}\, \underline{t}_i.$$

We transform relations (17.19)(a),(b); (17.21)(a),(b),(c); (17.23)(a),(b) and (17.25)(a) (if necessary) into equations over T_0. Let B_4 denote the T_0-algebra corresponding to this system of equations. Then:

(17.26) Proposition: Let $\varphi: B_4 \longrightarrow T$ be a T_0-algebra morphism satisfying:

(i) $\varphi(Q_\lambda) \in Q$ for all $\lambda \in \Lambda$

(ii) $\varphi(S) \notin (\varphi(V_1), \ldots, \varphi(V_e))T$

(iii) $\varphi(C) \equiv c \mod Q$

Then the T-ideal $(\varphi(Q_\sigma))_{\sigma\in\Sigma}T$ has a minimal prime divisor P_φ with properties:

(a) $Q \subseteq P_\varphi$

(b) ht $P_\varphi = \ell$

(c) $(T/Q)_{P_\varphi}$ is singular

(d) $W_i \nsubseteq P_\varphi$ for all $1 \le i \le r$.

<u>Proof</u>: As shown in (17.24) conditions (i) and (ii) guarantee (a), (b), (c). It remains to show (d). Let I denote the ideal generated by $\{\varphi(Q_\sigma)\}_{\sigma\in\Sigma}$ and suppose that $W_i \subseteq P_\varphi$ for some $1 \le i \le r$. Since $c \in W_i$ and $Q \subseteq P_\varphi$, (iii) implies that $\varphi(C) \in P_\varphi$. On the other hand by (17.25)(a) ht$(I, \varphi(C)) \ge \ell+1$, contradicting ht $P_\varphi = \ell$ and $(I, \varphi(C)) \subseteq P_\varphi$. Thus (d) holds.

§3: PROOF OF THEOREM (17.11)

We have to show:

(∗) Let (R,m) be a local domain with approximation property. Then for all $P \in \mathrm{Sing}(\hat{R})$: $P \cap R \neq 0$.

If R contains a field, let P denote the prime field contained in R. In the unequal characteristic case R contains a copy of $\mathbb{Z}_{(p)}$ where $p \in \mathbb{Z}$ is prime. In this case set $P = \mathbb{Z}_{(p)}$.

By Cohen's structure theorems $\hat{R} \cong W[\![x_2, \ldots, x_n]\!]/Q$ where W is a field or a Cohen ring. Note that $\hat{R}$ is a domain and thus $Q \subseteq W[\![x_2, \ldots, x_n]\!]$ a prime ideal. Let $x_1, x_2, \ldots, x_n$ be a system of generators of m_R with $x_1 = x_2$ in the equal characteristic case and $x_1 = p$ in the unequal characteristic case. Consider the natural maps:

$$R \xrightarrow{\gamma} \hat{R} \xrightarrow[\sim]{\delta} W[\![x_2, \ldots, x_n]\!]/Q.$$

This yields a commutative diagram:

$$\begin{array}{ccc} P[x_2, \ldots, x_n]_{(x_1, \ldots, x_n)} & \xrightarrow{\ \psi\ } & W[\![x_2, \ldots, x_n]\!] \\ \sigma \downarrow & & \downarrow \varsigma \\ R & \xrightarrow{\ \gamma\ } & \hat{R} \cong W[\![x_2, \ldots, x_n]\!]/Q \end{array}$$

(17.27) Set $T_0 = P[x_2, \ldots, x_n]_{(x_1, x_2, \ldots, x_n)}$ and $T = W[\![x_2, \ldots, x_n]\!]$. The morphism $\psi : T_0 \to T$ is faithfully flat and regular, hence ψ satisfies the conditions of section 2: By (17.4) the regular morphism $\psi: T_0 \to T$ is a direct limit of smooth T_0-algebras of finite type. In the following we identify T/Q and $\hat{R}$.

(17.28) Suppose that Theorem (17.11) is wrong. Then there is a minimal prime ideal $P \in \mathrm{Sing}(\hat{R})$ of minimal height with $P \cap R = 0$. Let P also denote the preimage of P in T.

<u>Case</u> 1: P is a prime ideal of minimal height in $\mathrm{Sing}(\hat{R})$. In this case let B be the finite type T_0-algebra B_3 from (17.23).
<u>Case</u> 2: P is not of minimal height in $\mathrm{Sing}(\hat{R})$. Let $\{W_1, \dots, W_r\}$ be the set of minimal prime ideals of $\mathrm{Sing}(\hat{R})$ with $\mathrm{ht}\, W_i < \mathrm{ht}\, P$. By assumption $\bigcap_{i=1}^{r} W_i \cap R \neq 0$ and we take $c_0 \in \bigcap_{i=1}^{r} W_i \cap R$. Let $c \in T$ be a preimage of c_0 in T. Since $P \cap R = 0$, $c \notin P$. In this case B denotes the finite type T_0-algebra B_4 with c chosen as above for the relation (17.25)(a).

(17.29) If we read the equations of section 2 as relations in T we obtain a natural T_0-algebra morphism $\tau: B \longrightarrow T$. By Theorem (17.4) τ factors through a smooth T_0-algebra D of finite type:

$$\begin{tikzcd} B \arrow[rr, "\tau"] \arrow[dr, "\eta"'] & & T \\ & D \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Let $Q_\sigma, V_i, S, Z, C,$ etc. denote the images of the variables in D under η.

(17.30) Set $E_0 = D \otimes_{T_0} R$. Note that E_0 is an R-algebra of finite type. φ induces an R-algebra morphism $\nu_0 : E_0 \longrightarrow \hat{R}$ satisfying the following properties:
(a) $\nu_0(Q_\lambda) = 0$ for all $\lambda \in \Lambda$
(b) $\nu_0(C) = c_0 \in R$.
Hence ν_0 factors through the R-algebra

$$E = E_0 / (Q_\lambda)_{\lambda \in \Lambda} + (C - c_0).$$

Let ν denote the R-algebra morphism $\nu : E \longrightarrow \hat{R}$.

(17.31) For all $n \in \mathbb{N}$ choose elements $s_n, q_{\sigma n} \in R$ so that

$$\nu(S) - s_n \in m^n \hat{R} \quad \text{and} \quad \nu(Q_\sigma) - q_{\sigma n} \in m^n \hat{R} \quad \text{for all } \sigma \in \Sigma - \Lambda.$$

Since R has the approximation property, for every $n \in \mathbb{N}$ there is an R-algebra morphism $\nu_n : E \longrightarrow R$ with the following properties:

(17.31)(a) $\quad \nu_n(S) - s_n \in m_R^n$

(17.31)(b) $\quad \nu_n(Q_\sigma) - q_{\sigma n} \in m_R^n$ for all $\sigma \in \Sigma - \Lambda$.

Moreover by the construction of E:

(17.31)(c) $\quad \nu_n(Q_\lambda) = 0$ for all $\lambda \in \Lambda$

(17.31)(d) $\quad \nu_n(C) = c_0$.

(17.32) Every ν_n induces a T_0-algebra morphism $\psi_n : D \longrightarrow R \subseteq \widehat{R} = T/Q$. Since D is smooth over T_0 and since T is a complete local ring, every ψ_n lifts to a T_0-algebra morphism $\varphi_n : D \longrightarrow T$. Thus for all $n \in \mathbb{N}$ there is a commutative diagram of T_0-algebra morphisms:

$$\begin{tikzcd} D \arrow[rr, "\psi_n"] \arrow[dr, "\varphi_n"'] & & R \subseteq \widehat{R} = T/Q \\ & T \arrow[ur, "\rho"'] & \end{tikzcd}$$

with properties:

(17.32)(a) $\quad \varphi_n(Q_\lambda) \in Q$ for all $\lambda \in \Lambda$

(17.32)(b) $\quad \varphi_n(C) \equiv c \mod Q$.

(17.33) <u>Claim:</u> $\varphi_n(S) \notin (\varphi_n(V_1), \text{—}, \varphi_n(V_e))$ for $n \in \mathbb{N}$ sufficiently large.

<u>Pf. of claim:</u> Suppose that $\varphi_n(S) \in (\varphi_n(V_1), \text{—}, \varphi_n(V_e)) \subseteq T$ for infinitely many n. Then $\psi_n(S) \in (\psi_n(V_1), \text{—}, \psi_n(V_e))$ and hence $\nu_n(S) \subseteq (\nu_n(V_1), \text{—}, \nu_n(V_e)) \subseteq (\nu_n(Q_\sigma))_{\sigma \in \Sigma}$ for infinitely many n. Hence by (17.31)(a) $\nu(S) \in P + m^n \widehat{R} \subseteq \widehat{R}$ for infinitely many n and therefore $\nu(S) \in P$. By construction of ν and φ: $\varphi(S) \notin P$, a contradiction.

(17.34) By (17.24) and/or (17.26) for all $n \geq n_0$ there is a prime ideal $P_n \subseteq T$ minimal over $(\psi_n(Q_\sigma))_{\sigma \in \Sigma} \subseteq T$ with the following properties:

(a) $\quad Q \subseteq P_n$

(b) $\operatorname{ht} P_n = \ell$

(c) $(T/Q)_{P_n}$ is singular

(d) $W_i \not\subseteq P_n$ for all $1 \le i \le r$, in the case that P_n is not of minimal height.

Let $\overline{P_n}$ be the image of P_n in $\widehat{R}$. Then $\overline{P_n} \in \operatorname{Sing}(\widehat{R})$ with $\overline{P_n} \cap R \neq 0$, since $\overline{P_n}$ is a minimal prime divisor of $(\nu_n(Q_\sigma))_{\sigma \in \Sigma}\widehat{R}$. Moreover, because $\operatorname{ht} \overline{P_n} = \operatorname{ht} P$ (in $\widehat{R}$) and because $\overline{P_n}$ does not contain any of the prime ideals $W_1, \ldots, W_r$, $\overline{P_n}$ is a minimal prime ideal of $\operatorname{Sing}(\widehat{R})$. Since $\operatorname{Sing}(\widehat{R})$ is closed in $\operatorname{Spec}(\widehat{R})$ there is an $n \in \mathbb{N}$ with $\overline{P_n} = \overline{P_{n+k}}$ for infinitely many $k \in \mathbb{N}$. By (17.31)(b) $P \subseteq \overline{P_n} + m^n\widehat{R}$ for all $n \in \mathbb{N}$. Thus $P \subseteq \overline{P_{n+k}} + m^{n+k}\widehat{R} = \overline{P_n} + m^{n+k}\widehat{R}$ for infinitely many $k \in \mathbb{N}$. This implies $P \subseteq P_n$ and by height reasons $P = P_n$.

Therefore $P \cap R \neq 0$.