CHAPTER II: ALGEBRAS WITH LIFTING PROPERTIES; THE ABSOLUTE CASE

§1: SMOOTH, UNRAMIFIED AND ÉTALE MORPHISMS

Let R be a commutative ring with identity and S an R-algebra via the morphism $\tau: R \to S$. For any R-algebra C and any ideal $I \subseteq C$ with $I^2 = 0$ let $\nu: C \to C/I$ denote the natural morphism. There is a canonical map of the sets of R-algebra morphisms:
$$\mathrm{Hom}_{R\text{-alg}}(S, C) \xrightarrow{\Phi} \mathrm{Hom}_{R\text{-alg}}(S, C/I)$$
defined by $\Phi(\varphi) = \nu\varphi$. We are interested in R-algebras S for which Φ is surjective, injective, and bijective for all R-algebras C and all ideals $I \subseteq C$ with $I^2 = 0$.
A different way of thinking about this property is by considering the following commutative diagram:
$$\begin{tikzcd} S \arrow[r, "\overline{\varphi}"] \arrow[dr, dashed, "\varphi"] & C/I \\ R \arrow[u, "\tau"] \arrow[r] & C \arrow[u, "\nu"'] \end{tikzcd}$$
Of interest are the R-algebras S for which whenever there is given such a commutative square (where $I \subseteq C$ is an ideal with $I^2 = 0$), any R-algebra morphism $\overline{\varphi}$ has a lifting φ, has at most one lifting φ, has a unique lifting φ, respectively. Define:

(2.1) Definition: Let R and S be as above.
(a) S is called a smooth R-algebra if for every R-algebra C and every ideal $I \subseteq C$ with $I^2 = 0$ the canonical map
$$\mathrm{Hom}_{R\text{-alg}}(S, C) \xrightarrow{\Phi} \mathrm{Hom}_{R\text{-alg}}(S, C/I)$$
with $\Phi(\varphi) = \nu\varphi$ is surjective.
(b) S is called formally unramified over R (or S is a formally unramified R-algebra if for every R-algebra C and every ideal $I \subseteq C$ with $I^2 = 0$ the map: $\mathrm{Hom}_{R\text{-alg}}(S, C) \xrightarrow{\Phi} \mathrm{Hom}_{R\text{-alg}}(S, C/I)$ is injective.

(c) An R-algebra S of finite type is called <u>unramified</u> if S is formally unramified over R.

(d) An R-algebra S is called étale over R if S is smooth and unramified over R, or equivalently, if S is of finite type over R and Φ is bijective for all R-algebras C and all ideals $I \subseteq C$ with $I^2 = 0$.

(2.2) <u>Exercise</u>: Let S be an R-algebra. Show that S is smooth (or formally unramified, étale) over R if and only if for every R-algebra C and every ideal $I \subseteq C$ with $I^n = 0$ for some $n \in \mathbb{N}$, the map

$$\mathrm{Hom}_{R\text{-alg}}(S, C) \xrightarrow{\Phi} \mathrm{Hom}_{R\text{-alg}}(S, C/I)$$

with $\Phi(\varphi) = \nu\varphi$, is surjective (or injective, bijective, respectively). In the étale case assume additionally that S is of finite type over R.

(2.3) <u>Proposition</u>: Let $\rho: R \longrightarrow S$ be a surjective morphism of rings. Then S is unramified over R.

<u>Proof</u>: Let C be an R-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Consider a commutative diagram of ring morphisms:

$$\begin{tikzcd} S \arrow[r, "\bar{u}"] \arrow[dr, shift left, "w"] \arrow[dr, shift right, "u"'] & C/I \\ R \arrow[u, "\rho"] \arrow[r, "\lambda"'] & C \arrow[u, "\nu"'] \end{tikzcd}$$

Suppose that u and w are liftings of $\bar{u}$. Then $u\rho = w\rho = \lambda$ and $u = w$ since ρ is surjective.

(2.4) <u>Proposition</u>: Let $\rho: R \longrightarrow S$ and $\tau: S \longrightarrow T$ be morphisms of rings. If S is smooth (or formally unramified, unramified, étale) over R and T is smooth (formally unramified, unramified, étale, resp.) over S, then T is smooth (formally unramified, unramified, étale) over R.

<u>Proof</u>: We only show the smooth and the formally unramified case. Let C be an R-algebra and $I \subseteq C$ an ideal with $I^2 = 0$.

(a) The smooth case:

Consider the commutative diagram of morphisms:

$$\begin{tikzcd}
T \arrow[rr, "\bar{u}"] & & C/I \\
S \arrow[u, "\tau"] & & \\
R \arrow[u, "\rho"] \arrow[rr, "\gamma"] & & C \arrow[uu, "\nu"']
\end{tikzcd}$$

By assumption S is smooth over R. Thus the R-algebra morphism $\bar{u}\tau$ lifts to an R-algebra morphism $v: S \longrightarrow C$ yielding a commutative diagram:

$$\begin{tikzcd}
T \arrow[r, "\bar{u}"] & C/I \\
S \arrow[u, "\tau"] \arrow[r, "v"] & C \arrow[u, "\nu"']
\end{tikzcd}$$

Since T is smooth over S, the S-algebra morphism $\bar{u}$ lifts to an S-algebra morphism $u: T \longrightarrow C$. Moreover, $u\tau\rho = \gamma$ and u is an R-algebramorphism lifting $\bar{u}$.

(b) <u>The formally unramified case</u>:

Consider the commutative diagram of ring morphisms:

$$\begin{tikzcd}
T \arrow[rr, "\bar{u}"] \arrow[ddrr, shift left, "u"] \arrow[ddrr, shift right, "v"'] & & C/I \\
S \arrow[u, "\tau"] & & \\
R \arrow[u, "\rho"] \arrow[rr, "\gamma"] & & C \arrow[uu, "\nu"']
\end{tikzcd}$$

where u and v lift $\bar{u}$. Since S is formally unramified and $u\tau$ and $v\tau$ both are liftings of $\bar{u}\tau$, we obtain that $u\tau = v\tau$. Consider C as a S-algebra via $u\tau = v\tau$. Since T is formally unramified over S it follows that $u = v$.

(2.5) <u>Proposition:</u> (Base change) Let R be a commutative ring, S and T R-algebras, and suppose that S is smooth (formally unramified, unramified, étale) over R. Then

(a) $S \otimes_R T$ is smooth (formally unramified, unramified, étale) over T.

(b) If additionally T is smooth (formally unramified, unramified, étale) over R, then $S \otimes_R T$ is smooth (formally unramified, unramified, étale) over R.

<u>Proof:</u> (a) We only prove the smooth and the formally unramified case. Let C be a T-algebra and $I \subseteq C$ an ideal with $I^2 = 0$. Consider a commutative diagram of ring morphisms:

$$\begin{tikzcd} S \arrow[r, "\beta"] & S \otimes_R T \arrow[r, "\bar{u}"] & C/I \\ R \arrow[u, "\alpha"] \arrow[r, "\gamma"'] & T \arrow[u, "\delta"] \arrow[r, "\sigma"] & C \arrow[u, "v"'] \end{tikzcd}$$

where α, γ, σ are the structure morphisms of the R-(or T)-algebras, β and δ are the natural maps into the tensor product. Consider C as an R-algebra via $\sigma\gamma$.

If S is smooth over R there is an R-algebra morphism $w: S \to C$ with $vw = \bar{u}\beta$ and $w\alpha = \sigma\gamma$. Let $u = w \otimes \sigma$, then $vu\beta = vw = \bar{u}\beta$ and $vu\delta = v\sigma = \bar{u}\delta$. By the universal property of the tensor product we obtain that $\bar{u} = vu$. Hence u is a T-algebra lifting of $\bar{u}$.

In the formally unramified case let in the above diagram $u, v: S \otimes_R T \to C$ be T-algebra liftings of $\bar{u}$, i.e. $vu = vv = \bar{u}$ and $u\delta = v\delta = \sigma$. Since S is formally unramified over R it follows that $u\beta = v\beta$. Using the universal property of the tensor product again we see that $u = v$.

(b) follows from (2.4).

(2.6) <u>Proposition:</u> Let R be a commutative ring, S an R-algebra, and R' a faithfully flat R-algebra. If $S' = S \otimes_R R'$ is formally unramified (unramified, respectively) over R', then S is formally unramified (unramified) over R.

<u>Proof:</u> Let C be an R-algebra and $I \subseteq C$ an ideal with $I^2 = 0$. Let $\bar{C} = C/I$,

$C' = C \otimes_R R'$, and $\overline{C}' = \overline{C} \otimes_R R' = C'/I'$ where $I' = IC' \cong I \otimes_R R'$ is nilpotent. Consider a commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\overline{u}"] \arrow[dr, shift left, "w"] \arrow[dr, shift right, "u"'] & \overline{C} \\ R \arrow[u] \arrow[r] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where ν is the natural map and u and w are two liftings of $\overline{u}$. By tensoring with R' we have a commutative diagram of R'-algebra morphisms:

$$\begin{tikzcd} S' \arrow[r, "\overline{u}\otimes 1"] \arrow[dr, shift left, "w\otimes 1"] \arrow[dr, shift right, "u\otimes 1"'] & \overline{C}' \\ R' \arrow[u] \arrow[r] & C' \arrow[u, "\nu'"'] \end{tikzcd}$$

Since S' is formally unramified over R', $u\otimes 1 = w \otimes 1$, and since R' is faithfully flat over R, $u = w$.

In the unramified case it remains to show that S is of finite type over R, if S' is of finite type over R'. This follows from the next Lemma:

(2.7) <u>Lemma</u>: Let R be a commutative ring, R' a faithfully flat R-algebra and S an R-algebra. If $S' = S \otimes_R R'$ is an R'-algebra of finite type, then S is an R-algebra of finite type.

<u>Proof</u>: Write $S = \varinjlim_{i\in I} S_i$, where $\{S_i\}_{i\in I}$ is the set of all finitely generated R-subalgebras of S. With $S_i' = S_i \otimes_R R'$, $S' = \varinjlim_{i\in I} S_i'$. Since R' is flat over R, the R'-algebras S_i' are contained in S'. Since S' is of finite type over R' there is a $k \in I$ with $S_i' = S'$ for all $i \in I$ with $k \leq i$. Since R' is faithfully flat over R, $S_i = S$ for all $k \leq i$ and S is finitely generated over R.

(2.8) <u>Proposition</u>: Let R be a ring, S an R-algebra, and $W \subseteq S$ a multiplicatively closed subset of S. Then

(a) $S_W = (W^{-1}S)$ is smooth over S.

(b) If S is formally unramified over R then S_W is formally unramified over R.

(c) Suppose that $W = \{1, f, f^2, \dots\}$ is generated by one element $f \in S$ and that $S_W = S_f$ is unramified (étale) over R. Then S_f is unramified (étale) over R.

<u>Proof</u>: (a) Let C be an S-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Consider the commutative diagram:

$$\begin{tikzcd} S_W \arrow[r, "\bar{u}"] & C/I \\ S \arrow[u] \arrow[r, "\tau"] & C \arrow[u, "v"'] \end{tikzcd}$$

For all $w \in W$ the image $v\tau(w) \in C/I$ is a unit. Since $I^2 = 0$, $\tau(w)$ is not contained in any prime ideal of C and $\tau(w)$ is a unit in C. τ factors through S_W.

(b) Let C be an R-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Consider the commutative diagram:

$$\begin{tikzcd} S \arrow[r, "i"] & S_W \arrow[r, "\bar{u}"] \arrow[dr, shift left, "w"] \arrow[dr, shift right, "u"'] & C/I \\ R \arrow[u] \arrow[rr, "\tau"] & & C \arrow[u, "v"'] \end{tikzcd}$$

where u and w are liftings of $\bar{u}$. Since S is formally unramified over R, $iu = iw$. Thus $u = w$.

(c) If S is of finite type over R, S_f is of finite type over R. By (b) S_f is unramified over R. If S is étale over R, S is smooth over R and S_f is smooth over S by (a). By (2.4) S_f is smooth over R.

(2.9) <u>Proposition</u>: Let R be a commutative ring, x a variable over R, $f \in R[x]$, and $f' \in R[x]$ its derivation. Set $S = R[x]/(f)$ and let $g \in R[x]$ be a polynomial with f' a unit in S_g. Then S_g is étale over R. In particular, $S_{f'}$ is étale over R.

<u>Proof</u>: Let C be an R-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Consider the commutative diagram of ring morphisms:

$$\begin{tikzcd} S_g \arrow[r, "\bar{u}"] & C/I \\ R \arrow[u] \arrow[r, "\tau"'] & C \arrow[u, "v"'] \end{tikzcd}$$

We have to show that $\bar{u}$ has a unique lifting $u : S \longrightarrow C$. Set $y = x + (f) \in S = R[y]$

and consider

$$\begin{tikzcd} S \arrow[r, "i"] & S_g & C \arrow[d, "\nu"] \\ & & C/I \arrow[from=1-1, "\bar{u}"'] \end{tikzcd}$$

with $\bar{c} = \bar{u}i(y)$ we claim that there is exactly one element $c \in C$ with $\nu(c) = \bar{c}$ and $f(c) = 0$. In order to show this let $c_0 \in C$ with $\nu(c_0) = \bar{c}$. Since $f(y) = 0$, it follows that $f(c_0) \in I$. Using Taylor's formula we obtain for every element $e \in I$ that $f(c_0+e) = f(c_0) + e\,f'(c_0) + q(e)$ where $q \in R[x]$ is a polynomial divisible by x^2. Since $I^2 = 0$, $q(e) = 0$ and $f(c_0+e) = f(c_0) + e\,f'(c_0)$. By assumption $f'(y) \in S_g$ is invertible, hence $f'(\bar{c}) \in C/I$ is invertible and $f'(c_0) \in C$ is invertible. Hence there is exactly one element $e \in I$ with $f(c_0+e) = 0$, namely, $e = -f(c_0)/f'(c_0)$.

By the claim there is exactly one morphism $\tilde{u}: S \longrightarrow C$ which lifts ui. The element $\tilde{u}(g)$ is invertible in C/I, thus $\tilde{u}(g)$ is invertible in C since I is nilpotent. Thus $\tilde{u}$ factors through S_g and S_g is étale over R.

(2.10) <u>Definition</u>: Let R be a commutative ring, $f \in R[x]$ a monic polynomial, and $g \in R[x]$ a polynomial such that f' is invertible in $S_g = (R[x]/(f))_g$. Then S_g is called a <u>standard étale</u> algebra over R.

(2.11) <u>Remark</u>: We will show later that any étale algebra is locally isomorphic to a standard étale algebra.

(2.12) <u>Exercise</u>: Let R be a commutative ring, x a variable over R. Show that the polynomial ring $R[x]$ is smooth over R.

§2. DIFFERENTIAL PROPERTIES

(2.13) Proposition: For an R-algebra S the following conditions are equivalent:

(a) S is formally unramified over R.

(b) $\Omega_{S/R} = 0$

Proof: (b)$\Rightarrow$(a): Let C be an R-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Consider a commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\bar{u}"] \arrow[dr, shift left, "w"] \arrow[dr, shift right, "u"'] & C/I \\ R \arrow[u] \arrow[r, "\tau"'] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where u and w are liftings of $\bar{u}$.

First note that u and w define the same S-module structure on I. Let $s \in S$ and $x \in I$, then $u(s)x - w(s)x = (u(s) - w(s))x = 0$ since $I^2 = 0$ and $(u-w)(s) \in I$.

We claim that $d = u - w : S \to I$ is an R-derivation from S into the S-module I. Since u and w are R-linear, d is R-linear. Let $s, t \in S$, then:

$$\begin{aligned} d(st) &= u(st) - w(st) \\ &= u(s)u(t) - w(s)u(t) + w(s)u(t) - w(s)w(t) \\ &= u(t)(u(s) - w(s)) + w(s)(u(t) - w(t)) \\ &= t\,d(s) + s\,d(t). \end{aligned}$$

Thus $d \in \mathrm{Der}_R(S, I) \cong \mathrm{Hom}_S(\Omega_{S/R}, I) = 0$ and $u = w$.

(a)$\Rightarrow$(b): We use the 'second' construction of the module of differentials. Let $\tau : S \otimes_R S \to S$ be defined by $\tau(s \otimes t) = st$ and let $J = \ker(\tau)$. Then $\Omega_{S/R} = J/J^2$.

Let $C = (S \otimes_R S)/J^2$. Obviously, $I = J/J^2$ is a nilpotent ideal of C and $C/J \cong S$.

Consider the commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\bar{u} = id"] \arrow[dr, shift left, "w"] \arrow[dr, shift right, "u"'] & C/I = S \\ R \arrow[u] \arrow[r, "\sigma"'] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where u and w are defined as follows: $u(s) = s \otimes 1 + J^2$ and $w(s) = 1 \otimes s + J^2$.

u and w are R-algebra liftings of $\bar{u}$. By (a) $u = w$ and $s \otimes 1 = 1 \otimes s$ for all $s \in S$

(in $\mathcal{J}/\mathcal{J}^2$). Thus $\mathcal{J}/\mathcal{J}^2 = 0$.

(2.14) Proposition: Let $k \to R \to S$ be morphisms of rings and suppose that S is smooth over R. Then the sequence of S-modules:

$$0 \longrightarrow \Omega_{R/k} \otimes_R S \longrightarrow \Omega_{S/k} \longrightarrow \Omega_{S/R} \longrightarrow 0$$

is exact.

Proof: By (1.17) we only have to show that $\alpha: \Omega_{R/k} \otimes_R S \longrightarrow \Omega_{S/k}$ is injective. By (1.16) this is equivalent to $0 \leftarrow \mathrm{Der}_k(R,M) \xleftarrow{\alpha^{**}} \mathrm{Der}_k(S,M)$ is exact for every S-module M. For an S-module M and $D \in \mathrm{Der}_k(R,M)$ consider the following commutative diagram of ring morphisms:

$$\begin{array}{ccc} S & \xrightarrow{id_S} & S \\ g \uparrow & & \uparrow \nu \\ R & \xrightarrow[\varphi]{} & S*M \end{array}$$

where:

(i) The ring $S*M$ is the trivial extension of S by the S-module M.

(ii) The map $\nu: S*M \longrightarrow S$ defined by $\nu(s,m) = s$ is the natural map onto $S*M/M \cong S$.

(iii) $\varphi: R \longrightarrow S*M$ is defined by $\varphi(r) = (g(r), D(r))$. Obviously, φ is a morphism of rings.

Since S is smooth over R there is an R-algebra morphism $h: S \longrightarrow S*M$ with $\nu h = id_S$. For every $t \in S$ write $h(t) = (t, \tilde{D}(t))$. This defines a map: $\tilde{D}: S \longrightarrow M$. For all $s,t \in S$: $h(st) = (st, \tilde{D}(st)) = h(s)h(t) = (s,\tilde{D}(s))(t,\tilde{D}(t)) = (st, s\tilde{D}(t) + t\tilde{D}(s))$ and $\tilde{D}$ is a k-derivation from S into M. For all $r \in R$: $D(r) = \tilde{D}(r)$ and α^{**} is surjective.

(2.15) Theorem: Let R be a commutative ring, S an R-algebra, and $I \subseteq S$ an ideal. If $\bar{S} = S/I$ is smooth over R, the sequence

$$0 \longrightarrow I/I^2 \xrightarrow{d} \Omega_{S/R} \otimes_S \bar{S} \longrightarrow \Omega_{\bar{S}/R} \longrightarrow 0$$

is split exact.

<u>Proof</u>: Consider the short exact sequence: $(*)\ 0 \longrightarrow I/I^2 \xrightarrow{\kappa} S/I^2 \xrightarrow{\mu} \overline{S} \longrightarrow 0$.

<u>Claim 1</u>: $(*)$ is split exact, more precisely, there is an R-algebra morphism $\varphi: \overline{S} \longrightarrow S/I^2$ with $\mu\varphi = id_{\overline{S}}$.

<u>Pf of Cl 1</u>: Consider the commutative diagram:

$$\begin{array}{ccc} \overline{S} & \xrightarrow{id_{\overline{S}}} & \overline{S} \\ \uparrow & & \uparrow \mu \\ R & \xrightarrow{\psi} & S/I^2 \end{array}$$

Since $\overline{S}$ is smooth over R, the identity map on $\overline{S}$ lifts to an R-algebra morphism $\varphi: \overline{S} \longrightarrow S/I^2$ with $\mu\varphi = id_{\overline{S}}$.

The composition map $\varphi\mu: S/I^2 \xrightarrow{\mu} S/I \xrightarrow{\varphi} S/I^2$ has the following properties:

(i) $\varphi\mu|_{I/I^2} = 0$

(ii) $\mu\,(id_{S/I^2} - \varphi\mu) = 0$.

Set $D = id_{S/I^2} - \varphi\mu : S/I^2 \longrightarrow S/I^2$. By condition (ii) $im(D) \subseteq I/I^2$.

<u>Claim 2</u>: D is an R-derivation.

<u>Pf of Cl.2</u>: Let $s, t \in S/I^2$, then:

$$\begin{aligned} D(st) &= st - (\varphi\mu)(st) \\ &= st - (\varphi\mu)(s)(\varphi\mu(t)) \\ &= st - s(\varphi\mu)(t) + s(\varphi\mu)(t) - (\varphi\mu)(s)(\varphi\mu)(t) \\ &= s(t - (\varphi\mu)(t)) + (\varphi\mu)(t)(s - (\varphi\mu)(s)) \end{aligned}$$

Note that $s - (\varphi\mu)(s) \in I/I^2$, thus $(t - (\varphi\mu)(t))(s - (\varphi\mu)(s)) = 0$ in S/I^2 and $(\varphi\mu)(t)(s - (\varphi\mu)(s)) = t\,(s - (\varphi\mu)(s))$. Thus D is a derivation. Since φ and μ are R-algebra morphisms, it also follows that $D|_R = 0$.

In order to prove the theorem we need to find an $\overline{S}$-linear map

$$v: \Omega_{S/R} \otimes_S \overline{S} \longrightarrow I/I^2$$

with $vd = id_{I/I^2}$. It suffices to show that the induced map:

$$Hom_{\overline{S}}(\Omega_{S/R} \otimes_S \overline{S}, I/I^2) \xrightarrow{d^*} Hom_{\overline{S}}(I/I^2, I/I^2)$$

defined by $d^*(\tau) = \tau d$, is surjective.

First note that $Hom_{\overline{S}}(\Omega_{S/R} \otimes_S \overline{S}, I/I^2) \cong Der_R(S, I/I^2)$. For every $\sigma \in Hom_{\overline{S}}(I/I^2, I/I^2)$ consider the composition map:

$$D': S \xrightarrow{\nu} S/I^2 \xrightarrow{D} I/I^2 \xrightarrow{\sigma} I/I^2$$

where ν is the natural map and D is the R-derivation from Claim 2. Then $D' \in \mathrm{Der}_R(S, I/I^2)$ corresponds to a linear map $\tau \in \mathrm{Hom}_{\bar{S}}(\Omega_{S/R} \otimes_S \bar{S}, I/I^2)$.

Claim 3: $d^*(\tau) = \tau d = \sigma$, where $\tau \in \mathrm{Hom}_{\bar{S}}(\Omega_{S/R} \otimes_S \bar{S}, I/I^2)$ as above.

Pf of Cl. 3: Let $\delta: S \longrightarrow \Omega_{S/R}$ be the universal R-derivation of S. Then $d: I/I^2 \longrightarrow \Omega_{S/R} \otimes_S \bar{S}$ is defined by $d(x+I^2) = \delta(x) + I\Omega_{S/R}$. Let $\bar{\delta}: S \longrightarrow \Omega_{S/R} \otimes_S \bar{S}$ denote the composition $S \xrightarrow{\delta} \Omega_{S/R} \xrightarrow{\lambda} \Omega_{S/R} \otimes_S \bar{S}$ where λ is the natural map. Consider the diagram:

$$\begin{tikzcd} & S \arrow[r, "D'"] \arrow[d, "\bar{\delta}"'] & I/I^2 \\ I/I^2 \arrow[r, "d"] & \Omega_{S/R} \otimes_S \bar{S} \arrow[ur, "\tau"'] & \end{tikzcd}$$

Then for all $x + I^2 \in I/I^2$:

$$\begin{aligned} \tau d(x+I^2) &= \tau(\delta(x) + I\Omega_{S/R}) \\ &= D'(x) && \text{by definition of } \tau \text{ and } D' \\ &= \sigma D(x+I^2) \\ &= \sigma((x+I^2) - (\varphi\mu)(x+I^2)) \\ &= \sigma(x+I^2) && \text{since } \varphi\mu|_{I/I^2} = 0 \text{ by (i).} \end{aligned}$$

Thus d^* is surjective and the assertion follows.

(2.16) Exercise: Let k be a ring, R a k-algebra, and $X = x_1, \ldots, x_n$ variables over R. Show:

$$\Omega_{R[X]/k} \cong (\Omega_{R/k} \otimes_R R[X]) \oplus \left(\bigoplus_{i=1}^{n} R[X]\delta x_i\right)$$

where $\delta: R[X] \longrightarrow \Omega_{R[X]/k}$ is the universal R-derivation.

§3: APPLICATIONS TO FIELD EXTENSIONS

(2.16) Proposition: Let $K \subseteq L$ be an algebraic field extension. Then:

(a) If L is separable over K, then $\Omega_{L/K} = 0$ and L is formally unramified over K.

(b) If L is finite and unramified over K, then L is separable over K.

Proof: Apply (1.21), (1.23), and (2.13).

(2.17) Lemma: Let $K \subseteq L$ be a finite separable field extension. Then L is smooth over K.

Proof: By separability $L = K(\alpha) = K[x]/(f)$ where $f \in K[x]$ is the minimal polynomial of α over K. Let C be a K-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Consider a commutative diagram:

$$\begin{tikzcd} L = K(\alpha) \arrow[r, "\bar{u}"] & C/I \\ K \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

Then $\bar{u}(f(\alpha)) = f(\bar{u}(\alpha)) = 0$. Let $c_0 \in C$ be a preimage of $\bar{u}(\alpha)$, then $f(c_0) \in I$. By Taylor's formula for all $y \in I$:

$$f(c_0 + y) = f(c_0) + y f'(c_0) + y^2 g$$

for some $g \in C$. Since $I^2 = 0$, $f(c_0+y) = f(c_0) + y f'(c_0)$. L is separable over K, thus $f'(\alpha) \neq 0$ and $\bar{u}(f'(\alpha))$ is a unit in C/I. Hence $f'(c_0)$ is a unit in C. With $e = -f(c_0)/f'(c_0) \in I$ we have that $f(c_0 + e) = 0$ in C. Thus $\bar{u}$ lifts to a K-algebra morphism $u : L \longrightarrow C$.

(2.18) Lemma: Let k be a ring, $\underline{x} = \{x_i\}_{i \in I}$ a set of variables over k. Then:

(a) The polynomial ring $k[\underline{x}]$ is smooth over k.

(b) If k is a field then the field of quotients $k(\underline{x})$ is smooth over k.

Proof: (a) follows from the universal property of polynomial rings and (b) is by (2.4) and (2.8).

(2.19) <u>Proposition</u>: Let $K \subseteq L$ be a finitely generated field extension. If L is separable over K, then L is smooth over K.

<u>Proof</u>: By (1.31) L is separably generated over K. Let $x_1, —, x_r$ be a separating transcendence basis of L over K, then $L = K(x_1, —, x_r, y)$ where y is separable algebraic over $K(x_1, —, x_r)$. By (2.17) L is smooth over $K(x_1, —, x_r)$ and by (2.18) $K(x_1, —, x_r)$ is smooth over K. The assertion follows by (2.4).

(2.20) <u>Theorem</u>: Let $K \subseteq L$ be a field extension. The following are equivalent:

(a) L is separable over K.

(b) L is smooth over K.

<u>Proof</u>: (b) $\Rightarrow$ (a): By (2.14) there is an exact sequence:

$$0 \longrightarrow \Omega_K \otimes_K L \longrightarrow \Omega_L \longrightarrow \Omega_{L/K} \longrightarrow 0.$$

By (1.47) L is separable over K.

(a) $\Rightarrow$ (b): Let C be a K-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Set $\overline{C} = C/I$ and consider a commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\overline{u}} & \overline{C} \\ \uparrow & & \uparrow \\ K & \longrightarrow & C \end{array}$$

<u>Case 1</u>: char $K = 0$

<u>Pf</u>: By Zorn's Lemma there is a maximal intermediate field $K \subseteq E \subseteq L$ such that $\overline{u}|_E$ lifts to a K-algebra morphism u_0 from E to C. Suppose that $E \neq L$ and let $x \in L - E$. Since char $K = 0$, $E(x)$ is separable and thus smooth over E. Hence u_1 extends to an E-algebra morphism $u_1: E(x) \longrightarrow C$ which lifts $\overline{u}|_{E(x)}$, a contradiction.

<u>Case 2</u>: char $K = p > 0$

<u>Pf</u>: Let B be a p-basis of L over K and set $E = K(B)$. By (1.48) we know:

(i) B is algebraically independent over K

(ii) L is separable over $E = K(B)$.

Thus there is a K-algebra morphism $u_0 : E \longrightarrow C$ so that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\overline{u}"] & \overline{C} \\ E \arrow[u] \arrow[dr, "u_0"] & \\ K \arrow[u] \arrow[r] & C \arrow[uu, "\mu"'] \end{tikzcd}$$

<u>Claim</u>: There is a K-algebra morphism $u_1 : L^p \longrightarrow C$ with $\mu u_1 = \overline{u}|_{L^p}$.

Pf of Cl: Consider the following subring $\overline{C}^p$ of $\overline{C}$:

$$\overline{C}^p = \{ y \in \overline{C} \mid y = z^p \text{ for some } z \in \overline{C} \}.$$

Since char $K = p > 0$, C is a ring of characteristic $p > 0$ and there is a map $g : \overline{C}^p \longrightarrow C$ defined as follows: For every $y = z^p \in \overline{C}^p$ let z_0 be preimage of z in C. With $g(y) = z_0^p$, g is well defined since $I^2 = 0$. Note that g is a K-algebra morphism. Since $L^p \hookrightarrow \overline{C}^p$, $u_1 = g|_{L^p}$ is a lifting of $\overline{u}|_{L^p}$.

In order to finish the proof consider the morphism:

$$u_1 \otimes u_0 : L^p \otimes_{K(B)^p} E \longrightarrow C.$$

Note that $u_1 \otimes u_0$ is well defined since $u_1|_{K(B)^p} = u_0|_{K(B)^p}$. By assumption L is separable over K and E. By (1.45) L and $E^{p^{-1}}$ are linearly disjoint over E, or equivalently, L^p and E are linearly disjoint over $E^p = K(B)^p$. Hence

$$L^p \otimes_{E^p} E \cong L^p[E]$$

and u_1 extends to a morphism $u_2 : L^p[E] \longrightarrow C$ with $\mu u_2 = \overline{u}|_{L^p[E]}$. Then $L = Q(L^p[E])$, the quotient field of $L^p[E]$, and u_2 extends to a K-algebra morphism $u : L \longrightarrow C$ with $\mu u = \overline{u}$.

(2.21) <u>Definition</u>: Let k be a field, and $k \longrightarrow A \longrightarrow B$ morphisms of rings. The B-module

$$\Gamma_{B/A/k} = \ker\left(\Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k}\right)$$

where α is the canonical map, is called the <u>module of imperfection</u> of A and B over k. If k is a prime field, we write $\Gamma_{B/A}$.

(2.22) <u>Lemma</u>: Let $k \hookrightarrow K \hookrightarrow L \hookrightarrow L'$ be field extensions. Then there is an exact sequence:

$$0 \longrightarrow \Gamma_{L/K/k} \otimes_L L' \longrightarrow \Gamma_{L'/K/k} \longrightarrow \Gamma_{L'/L/k} \longrightarrow \Omega_{L/K} \otimes_L L' \longrightarrow$$
$$\longrightarrow \Omega_{L'/K} \longrightarrow \Omega_{L'/L} \longrightarrow 0.$$

<u>Proof</u>: We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Gamma_{L/K/k} \otimes_L L' & \longrightarrow & \Omega_{K/k} \otimes_K L' & \longrightarrow & \Omega_{L/k} \otimes_L L' & \longrightarrow & \Omega_{L/K} \otimes_L L' & \longrightarrow 0 \\
& \downarrow f_1 & & \downarrow \mathrm{id} & & \downarrow f_2 & & \downarrow f_3 & \\
0 \longrightarrow & \Gamma_{L'/K/k} & \longrightarrow & \Omega_{K/k} \otimes_K L' & \longrightarrow & \Omega_{L'/k} & \longrightarrow & \Omega_{L'/K} & \longrightarrow 0
\end{array}$$

where f_2 and f_3 are the canonical maps. Abbreviate the diagram to a commutative diagram with exact rows (of vector spaces):

$$\begin{array}{ccccccccc}
0 \longrightarrow & X & \longrightarrow & A & \longrightarrow & B & \longrightarrow & P & \longrightarrow 0 \\
& \downarrow f_1 & & \downarrow \mathrm{id} & & \downarrow f_2 & & \downarrow f_3 & \\
0 \longrightarrow & Y & \longrightarrow & A & \longrightarrow & C & \longrightarrow & Q & \longrightarrow 0
\end{array}$$

f_1 is injective, hence we may assume $X \subseteq Y$ where X and Y are subspaces of A. This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & A/X & \longrightarrow & B & \longrightarrow & P & \longrightarrow 0 \\
& \nu \downarrow & & \downarrow f_2 & & \downarrow f_3 & \\
0 \longrightarrow & A/Y & \longrightarrow & C & \longrightarrow & Q & \longrightarrow 0
\end{array}$$

where ν is the natural map. By the Snake Lemma we obtain an exact sequence:

$$0 \longrightarrow \ker(\nu) = Y/X \longrightarrow \ker(f_2) \longrightarrow \ker(f_3) \longrightarrow \mathrm{coker}(\nu) = 0$$

This induces an exact sequence:

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \ker(f_2) \longrightarrow P \longrightarrow Q \longrightarrow \mathrm{coker}(f_3) \longrightarrow 0$$

proving the Lemma.

(2.23) <u>Theorem</u>: (The Cartier equality) Let k be a perfect field, $k \subseteq K \subseteq L$ field extensions with L finitely generated over K. Then

$$\mathrm{rk}_L(\Omega_{L/K}) = \mathrm{trdeg}_K L + \mathrm{rk}_L(\Gamma_{L/K/k}).$$

Proof: We first prove:

Claim: Let $k \subseteq K \subseteq E \subseteq L$ be field extensions where E and L are finitely generated over K. If the Cartier equality holds for $k \subseteq E \subseteq L$ and for $k \subseteq K \subseteq E$, then the Cartier inequality holds for $k \subseteq K \subseteq L$.

Pf of Cl: By assumption: $\quad \mathrm{rk}_L(\Omega_{L/E}) = \mathrm{trdeg}_E L + \mathrm{rk}_L(\Gamma_{L/E/k})$ and

$$\mathrm{rk}_E(\Omega_{E/k}) = \mathrm{trdeg}_K E + \mathrm{rk}_E(\Gamma_{E/K/k}).$$

By (2.22) there is an exact sequence:

$$0 \longrightarrow \Gamma_{E/K/k} \otimes_E L \longrightarrow \Gamma_{L/K/k} \longrightarrow \Gamma_{L/E/k} \longrightarrow \Omega_{E/k} \otimes_E L \longrightarrow$$
$$\longrightarrow \Omega_{L/k} \longrightarrow \Omega_{L/E} \longrightarrow 0.$$

This implies:

$$\begin{aligned} \mathrm{rk}_L(\Omega_{L/K}) - \mathrm{rk}_L(\Gamma_{L/K/k}) &= \mathrm{rk}_L(\Omega_{E/k} \otimes_E L) + \mathrm{rk}_L(\Omega_{L/E}) - \\ &\quad - \mathrm{rk}_L(\Gamma_{E/K/k} \otimes_E L) - \mathrm{rk}_L(\Gamma_{L/E/k}) \\ &= \mathrm{trdeg}_K E + \mathrm{trdeg}_E L \\ &= \mathrm{trdeg}_K L. \end{aligned}$$

This proves the claim.

By assumption the field L is obtained from K by a finite succession of field extensions of the following type:

(a) $L = K(\alpha)$, α is transcendental over K

(b) $L = K(\alpha)$, α is separable algebraic over K

(c) $L = K(\alpha)$, $\mathrm{char}\, K = p > 0$, $\alpha \notin K$, and $\alpha^p \in K$.

By the claim we only need to prove the Cartier equality in cases (a), (b), and (c).

In cases (a) and (b) L is smooth over K and by (2.14) there is an exact sequence:

$$0 \longrightarrow \Omega_{K/k} \otimes_K L \longrightarrow \Omega_{L/k} \longrightarrow \Omega_{L/K} \longrightarrow 0.$$

In particular, $\Gamma_{L/K/k} = 0$. In (a) $\Omega_{L/K} \cong L$ and in (b) $\Omega_{L/K} = 0$ yielding Cartier's equality in those cases.

In the last case $L \cong K[x]/(x^p - a)$. Since k is perfect, L is separable and hence smooth over k. By (2.15) there is an exact sequence:

$$0 \longrightarrow (x^p-a)/(x^p-a)^2 \xrightarrow{\delta} \Omega_{K[x]/k} \otimes_{K[x]} L \longrightarrow \Omega_{L/k} \longrightarrow 0.$$

Thus $\quad \Omega_{L/k} = (\Omega_{K[x]/k} \otimes_K L)/\mathrm{im}(\delta)$

$\qquad\quad = (\Omega_{K[x]/k} \otimes_{K[x]} L)/L\delta(a)$

By (2.16)

$$\Omega_{K[x]/k} \cong (\Omega_{K/k} \otimes_K K[x]) \oplus K[x]\delta_{K[x]}(x)$$

This implies:

$$(*) \qquad \Omega_{L/k} \cong ((\Omega_{K/k}/K\delta_K(a)) \otimes_K L) \oplus L\delta_L(\alpha)$$

where δ_K and δ_L are the universal k-derivations of K and L.

Consider the exact sequence:

$$0 \longrightarrow \Gamma_{L/K/k} \longrightarrow \Omega_{K/k} \otimes_K L \xrightarrow{\tau} \Omega_{L/k} \longrightarrow \Omega_{L/K} \longrightarrow 0$$

Since k is perfect, $a \notin K^p(k) = K^p$ and $\{a\}$ extends to a p-basis $\{a\} \cup B$ of K over k. By $(*)$ $\tau(\delta_K(a)) = 0$ and $\tau(\delta_K(B))$ is linearly independent in $\Omega_{K/k}$. Moreover, $\delta_L(\alpha) \cup \tau(\delta_K(B))$ generates $\Omega_{L/k}$. This shows $\mathrm{rk}_L(\Gamma_{L/K/k}) = 1$. Since α is p-independent over k ($L^p(k) = L^p \subseteq K$), $\{\alpha\}$ is part of a p-basis of L over k. Thus $\delta_L(\alpha) \neq 0$ and $\mathrm{rk}_L(\Omega_{L/K}) = 1$, as desired.