CHAPTER III: THE STRUCTURE OF UNRAMIFIED AND ETALE MORPHISMS

§1: QUASI FINITE EXTENSIONS

(3.1) Proposition: Let $\varphi: R \to S$ be a homomorphism of rings, S an R-algebra of finite type via φ. Then the following conditions are equivalent:

(a) S is unramified over R.

(b) For all prime ideals $P \in \operatorname{Spec}(R)$ the fiber ring $S \otimes_R k(p)$ is unramified over $k(P)$.

Proof: (a) $\Rightarrow$ (b): Follows from (2.6).

(b) $\Rightarrow$ (a): We have to show that $\Omega_{S/R} = 0$, or equivalently, that $(\Omega_{S/R})_Q = 0$ for every prime ideal $Q \in \operatorname{Spec}(S)$. Let $Q \in \operatorname{Spec}(S)$ and $P = \varphi^{-1}(Q) \in \operatorname{Spec}(R)$. Then

$$(\Omega_{S/R})_Q = \Omega_{S_Q/R} = \Omega_{S_Q/R_P}.$$

Since S is a finitely generated R-algebra, $\Omega_{S/R}$ is a finitely generated S-module and $(\Omega_{S/Q})_Q$ is a finitely generated S_Q-module. We may assume that R is a quasi local ring with maximal ideal m, $Q \in \operatorname{Spec}(S)$ with $\varphi^{-1}(Q) = m$, and that $(\Omega_{S/R})_Q$ is a finitely generated S_Q-module. By (1.12) there is an exact sequence:

$$mS/m^2S \xrightarrow{d} \Omega_{S/R}/m\,\Omega_{S/R} \xrightarrow{\pi} \Omega_{\bar S/R} \longrightarrow 0$$

where d is induced by the universal R-derivation $\delta: S \to \Omega_{S/R}$. Since $m \subseteq R$, $\operatorname{im}(d) = 0$ and thus

$$\Omega_{\bar S/R} = \Omega_{\bar S/k} = \Omega_{S/R}/m\,\Omega_{S/R}.$$

Since $\bar S$ is unramified over k, $\Omega_{\bar S/k} = 0$. Hence

$$(\Omega_{S/R}/m\,\Omega_{S/R})_Q = (\Omega_{S/R})_Q/m\,(\Omega_{S/R})_Q = 0.$$

The assertion follows from Nakayama's Lemma.

(3.2) Remark: The proposition shows that in order to study unramified morphisms it suffices to study unramified morphisms over fields.

(3.3) Theorem: (Structure of unramified extensions over fields) Let K be a field, $\overline{K}$ its algebraic closure, and R a K-algebra of finite type. The following conditions are equivalent:

(a) R is unramified over K.

(b) $\Omega_{R/K} = 0$

(c) $R \cong \prod_{i=1}^{r} L_i$, where L_i are finite separable extension fields of K.

(d) $R \cong \prod_{i=1}^{t} K[x]/(f_i)$, where $f_i \in K[x]$ are monic polynomials with $(f_i, f_i') = 1$, i.e. f_i and f_i' are relatively prime in $K[x]$.

(e) $R \otimes_K \overline{K} \cong \prod_{i=1}^{n} \overline{K}$

(f) R is a finite separable K-algebra, that is, R is a finite K-algebra and for every extension field $K \subseteq L$ the tensor product $R \otimes_K L$ is reduced.

Proof: (a)$\Leftrightarrow$(b): By (2.13).

(c) $\Rightarrow$ (d): obvious

(d) $\Rightarrow$ (c): Write every polynomial f_i as a product of monic irreducible polynomials in $K[x]$: $f_i = \prod_{j=1}^{e_i} p_{ij}$. Since f_i and f_i' are relatively prime, p_{ij} and p_{ij}' are relatively prime for all j. In particular, for all i,j: p_{ij} is a separable polynomial. Then (c) follows from the Chinese Remainder Theorem.

(e) $\Rightarrow$ (b): Set $\overline{R} = R \otimes_K \overline{K}$. We first show that $\Omega_{\overline{R}/\overline{K}} = 0$. Let N be an $\overline{R}$-module and $d: \overline{R} \longrightarrow N$ a $\overline{K}$-derivation. For every element $e_i = (0, -, 0, 1, 0, -, 0) \in \prod_{i=1}^{n} \overline{K}$ in the standard $\overline{K}$-basis of $\overline{R}$ we have that $e_i = e_i^2$. Hence $d(e_i) = d(e_i^2) = 2e_i d(e_i)$, and $(1-2e_i)d(e_i) = 0$. Since $1-2e_i$ is a unit in $\overline{R}$, $d(e_i) = 0$ for all $1 \leq i \leq n$. Thus $d = 0$, in particular, $\Omega_{\overline{R}/\overline{K}} = 0$ and $\overline{R}$ is unramified over $\overline{K}$. By (2.6) R is unramified over K and $\Omega_{R/K} = 0$ by (2.13).

(d) $\Rightarrow$ (e): We may assume that $t = 1$, that is, $R = K[x]/(f)$, where $f \in K[x]$ is a monic polynomial with $(f, f') = 1$. Then $\overline{R} = R \otimes_K \overline{K} = \overline{K}[x]/(f)$ and f and f' are also relatively prime in $\overline{K}[x]$. Hence $f = \prod_{i=1}^{s} (x - \lambda_i)$ with $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Hence $\overline{R} = \prod_{i=1}^{s} \overline{K}$ by the Chinese Remainder Theorem.

(e) $\Rightarrow$ (d): We consider R as a subring of $\overline{R} = R \otimes_K \overline{K}$. Every element of $\overline{R}$ is integral over $\overline{K}$, hence integral over K. Thus R is integral over K. Since R is finitely generated over K, R is an Artinian ring and R is reduced since $\overline{R}$ is reduced. Thus R is a product of fields $R = \prod_{i=1}^{t} K_i$ and every K_i is a finite field extension of K. Moreover, for all $i = 1, \dots, t$ the ring $K_i \otimes_K \overline{K}$ is reduced. Thus each K_i is separable over K.

(d)$\Leftrightarrow$(f): by similar arguments

(b)$\Rightarrow$(e): By (2.5) $\overline{R} = R \otimes_K \overline{K}$ is unramified over $\overline{K}$. Let R be unramified over K where K is an algebraically closed field.

Case 1: R is a finite K-algebra.

Then R is an Artinian ring and thus $R \cong \prod_{i=1}^{s} R_i$, where R_i are local Artinian ring. Each R_i is a homomorphic image of R, thus each R_i is unramified over K (by (2.3) and (2.4)). Thus we may assume that R is a local Artinian ring with maximal ideal m. Since R is finite over the algebraically closed field K we have that $R/m = K$ and thus $R = K + m$. The tensor product $R \otimes_K R$ is a finitely generated K-module, hence $R \otimes_K R$ is an Artinian ring. Consider the canonical map: $\tau : K \cong K \otimes_K K \longrightarrow (R \otimes_K R)_{red}$. Every element $a \otimes b \in R \otimes_K R$ with $a \in m$ or $b \in m$ is nilpotent and τ is an isomorphism. Hence $R \otimes_K R$ is a local Noetherian ring.

We claim that $R \otimes_K R \cong R$. In order to prove this consider the short exact sequence: $0 \longrightarrow I \longrightarrow R \otimes_K R \xrightarrow{\mu} R \longrightarrow 0$ where μ is defined by $\mu(a \otimes b) = ab$ and $I = \ker(\mu)$. The ideal I is contained in the maximal ideal of R and by assumption $\Omega_{R/K} = 0$ and $\Omega_{R/K} \cong I/I^2$. Hence by Nakayama $I = 0$ and $R \otimes_K R \cong R$.

If $\dim_K(R) = n$ then $\dim_K(R \otimes_K R) = n^2$. Hence $n = 1$ and $R = K$.

Case 2: R is of finite type over K.

For every maximal ideal $m \subseteq R$ the residue field $k(m) = R/m$ is of finite type over K. By the Nullstellensatz $k(m)$ is finite over K and hence $K = k(m)$ since K is algebraically closed. Then, by induction on n, the rings R/m^n are finite over K for all $n \in \mathbb{N}$. By (2.3) and (2.4) for all $n \in \mathbb{N}$ the ring R/m^n is finite and unramified over K. By the first case $R/m^n = K$ for all $n \in \mathbb{N}$. Therefore $m = m^n$ for all $n \in \mathbb{N}$ and $mR_m = 0$. This shows that every maximal ideal of R is minimal and R is an Artinian ring. In particular, R is finite over K. By the first case R is a product of fields.

(3.4) Corollary: Let K be an infinite field and R an unramified K-algebra. Then

$$R \cong K[x]/(f)$$

where $f \in K[x]$ is a monic polynomial with $(f, f') = 1$.

Proof: By (3.3): $\quad R \cong \prod_{i=1}^{t} K[x]/(f_i)$

where f_i are monic polynomials in $K[x]$ with $(f_i, f_i') = 1$. We may assume that every f_i is irreducible in $K[x]$. Hence $K(\alpha_i) = K[x]/(f_i)$ for an element α_i in the algebraic closure $\overline{K}$ of K. Moreover, each α_i is separable over K. Since K is an infinite field there are infinitely many elements $\beta_i \in K(\alpha_i)$ with $K(\alpha_i) = K(\beta_i)$. By choosing the α_i appropriately we can achieve that $(f_i, f_j) = 1$ for $i \neq j$. Then, by the Chinese Remainder Theorem: $R \cong K[x]/(f)$ with $f = \prod_{i=1}^{t} f_i$ and $(f, f') = 1$.

(3.5) Remark: Theorem (3.3) shows that for an unramified R-algebra S all fibers of S over R are finite. In particular, for all $P \in \operatorname{Spec}(R)$ there are only finitely many prime ideals in S which lie over P.

(3.6) Proposition: Let K be a field and R a K-algebra of finite type. For every prime ideal $Q \in \operatorname{Spec}(R)$ the following conditions are equivalent:

(a) The set $\{Q\}$ is open in $\operatorname{Spec}(R)$.

(b) The ring R_Q is finite over K.

(c) Q is maximal and minimal in $\operatorname{Spec}(R)$.

<u>Proof</u>: (a) $\Rightarrow$ (b): For every $f \in R$ let D_f denote the basic open set $D_f = \{P \in \mathrm{Spec}(R) \mid f \notin P\}$. Note that D_f is homeomorphic to $\mathrm{Spec}(R_f)$. Since $\{Q\}$ is open, $\{Q\} = D_f$ for some $f \in R$ and $\mathrm{Spec}(R_f) = \{Q\}$. Therefore R_f is a local Artinian ring with maximal ideal QR_f. Since R_f is of finite type over K by the Nullstellensatz the field R_f/QR_f is finite over K. Hence $R_f = R_Q$ is finite over K.

(b) $\Rightarrow$ (c): By assumption (b) Q is a minimal prime ideal of R. Let $\varphi : R \longrightarrow R_Q$ be the natural morphism and $I = \ker(\varphi)$. Since R is Noetherian there is a $t \in R - Q$ with $t \in \mathrm{ann}(I)$ and $\varphi_t : R_t \longrightarrow R_Q$ is injective. Since R_Q is finite over K, R_t is finite over K and there is an element $s \in R_t$ with $R_{ts} = R_Q$. Therefore there is an $f \in R - Q$ with $R_f = R_Q$. Consider the morphism $\tau : K[x] \longrightarrow R$ defined by $\tau(x) = f$.

If $\ker(\tau) = 0$, then $\tau_x : K[x]_x \longrightarrow R_f$ is injective and R_f is not finite over K. Therefore $\ker(\tau) \neq 0$ and there is a minimal integer $n \in \mathbb{N}$ with

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \ker(\tau)$$

where $a_i \in K$ for all $0 \le i \le n-1$. Hence $f^n + a_{n-1}f^{n-1} + \ldots + a_0 = 0$ in R. If $a_0 \neq 0$, then f is invertible in R. In this case $R = R_f = R_Q$ and we are done. If $a_0 = 0$, let $k \in \{0, \ldots, n-1\}$ be minimal with $a_k = 0$. (Note that $f^n = 0$ implies that R_f is the null ring). Then $f^n + a_{n-1}f^{n-1} + \ldots + a_k f^k = 0$, in particular, $f^{n-k} + a_{n-1}f^{n-k-1} + \ldots + a_k \in Q$, since Q is prime. Suppose that $P \in \mathrm{Spec}(R)$ with $Q \subsetneq P$. Since $P \notin D_f$, $f \in P$ and hence $a_k \in P$, a contradiction.

(c) $\Rightarrow$ (a): Let $Q, Q_1, \ldots, Q_r$ be the minimal prime ideals of R. Then

$$\mathrm{Spec}(R) = V(Q_1 \cap \ldots \cap Q_r) \cup V(Q).$$

Since $V(Q) = \{Q\}$ the sets $V(Q_1 \cap \ldots \cap Q_r)$ and $V(Q)$ are disjoint and $\{Q\}$ is open in $\mathrm{Spec}(R)$.

(3.7) <u>Notation</u>: Let R be a ring, S an R-algebra and $\varphi : R \longrightarrow S$ the structure morphism. If $Q \in \mathrm{Spec}(S)$ we write $Q \cap R := \varphi^{-1}(Q)$, although the morphism φ may not be injective.

(3.8) Corollary: Let R be a ring, S an R-algebra of finite type. Let $Q \in \mathrm{Spec}(S)$ be a prime ideal and $P = Q \cap R \in \mathrm{Spec}(R)$ its preimage. Then the following conditions are equivalent:

(a) The set $\{Q\} = \{(Q/PS)_P\}$ is open in $\mathrm{Spec}(S \otimes_R k(P))$

(b) The ring $(S/PS)_Q$ is a finite $k(P)$-algebra

where S_P denotes the localization $T^{-1}S$ with $T = R - P$.

(3.9) Definition: Let R be a ring, S an R-algebra of finite type, $Q \in \mathrm{Spec}(S)$ a prime ideal and $P = Q \cap R$ its preimage in R.

(a) S is called quasi finite over R in Q if $(S/PS)_Q$ is finite over $k(P)$.

(b) S is called quasi finite over R if S is quasi finite over R in every prime ideal $Q \in \mathrm{Spec}(S)$.

(3.10) Proposition: Let R be a ring and S an R-algebra of finite type. The following conditions are equivalent:

(a) S is quasi finite over R.

(b) For every prime ideal $P \in \mathrm{Spec}(R)$ the ring $S \otimes_R k(P)$ is finite over $k(P)$.

Proof: (b) $\Rightarrow$ (a): obvious

(a) $\Rightarrow$ (b): Let $P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(S)$ with $P = Q \cap R$. Since $(S/PS)_Q$ finite over $k(P)$, Q is a minimal prime ideal of $S \otimes_R k(P)$. Since S is of finite type over R, $S \otimes_R k(P)$ is of finite type over $k(P)$ and therefore $S \otimes_R k(P)$ is an Artinian ring. Then $S \otimes_R k(P) = \prod_{i=1}^n S_i$ with local Artinian rings S_i. By assumption each $S_i \cong (S \otimes_R k(P))_Q$ is finite over $k(P)$ and hence $S \otimes_R k(P)$ is finite over $k(P)$.

(3.11) Corollary: Let R be a ring and S an unramified R-algebra. Then S is quasi finite over R.

§2: ZARISKI'S MAIN THEOREM

(3.12) Theorem: (Zariski's Main Theorem) Let R be a ring and S an R-algebra of finite type. Let R' denote the subring of S which consists of all elements of S which are integral over R and let $Q \subseteq S$ be a prime ideal, $P = Q \cap R$ its preimage in R. If S is quasi finite over R in Q, then there is an element $f \in R'$, $f \notin Q$, with $R'_f = S_f$.

We present a proof of Zariski's Main Theorem (ZMT) which is due to Peskine and Szpiro. It is very long and consists of several lemmas and a proposition.

(3.13) Lemma: Let $R \to T \to S$ be morphisms of rings and $Q \subseteq S$ a prime ideal. If S is quasi finite over R in Q, then S is quasi finite over T in Q.

Proof: By definition (3.9) S is of finite type over R, hence S is of finite type over T. With $P = Q \cap R$ and $W = Q \cap T$ we have a commutative diagram of ring morphisms:

$$\begin{array}{ccc} k(P) & \xrightarrow{\text{inj.}} & k(W) \\ \downarrow & & \downarrow \\ (S/PS)_Q & \xrightarrow{\text{surj.}} & (S/WS)_Q \end{array}$$

Hence $(S/WS)_Q$ is a finite $k(W)$-vector space and the assertion follows.

(3.14) Proposition: Let $R \to S$ be an injective morphism of rings. Suppose that S is of finite type over R and that R is integrally closed in S. If S is quasi finite over R in Q for some $Q \in \mathrm{Spec}(S)$, then $R_P = S_P$ where $P = Q \cap R$ and $S_P = T^{-1}S$ with $T = R - P$.

(3.15) Lemma: Proposition (3.14) implies ZMT (3.12).

Proof: We have ring morphisms $R \xrightarrow{\lambda} R' \xrightarrow{\sigma} S$ with σ injective. Since S is quasi finite over R in Q, by (3.13) S' is quasi finite over R' in Q. Moreover, R' is integrally closed in S'. By (3.14) $R'_P = S_P$ where $P = Q \cap R'$. S' is an R-algebra of finite type, say $S' = R'[y_1, -, y_n]$. The natural map $\tau: S \to S_P = R'_P$ maps y_i into an element $b_i \in R'_P$. Let $a \in R' - P$ so that there are elements $c_i \in R'_a$ with $\sigma(c_i) = b_i$ where $\sigma: R'_a \to R'_P$ is the natural map. Consider R'_a as a subring of S'_a. Since $R'_P = S_P$ there is an element $t \in R' - P$ so that $c_i/1 = y_i/1$ in $(S_a)_t$. Therefore the map $(R_a)_t \to (S_a)_t$ is surjective and $R'_f = S_f$ for $f = at$.

(3.16) <u>Lemma</u>: Let $R \hookrightarrow R[x] \hookrightarrow S$ be ring extensions satisfying:

(i) S is finite over $R[x]$.

(ii) R is integrally closed in S.

If S is quasi finite over R in some $Q \in \mathrm{Spec}(S)$ then $S_P = R_P$ where $P = Q \cap R$.

We first show:

(3.17) <u>Lemma</u>: Lemma (3.16) implies Proposition (3.14).

<u>Proof of (3.17)</u>: Let $R \to S'$ be an injective morphism of rings and suppose that R is an integrally closed subring of S'. In particular, there is a subring T of R and ring extensions $R \hookrightarrow T \hookrightarrow S'$ so that T is of finite type over R and S is a finitely generated T-module. If $T = R[x_1, -, x_n]$ we proceed by induction on the number of generators of T.

If $n = 0$, S' is integral over R and R is integrally closed in S'. Hence $R = S'$. For the induction step set $C = R[x_1, -, x_{n-1}]$ and consider ring extensions:

$$R \hookrightarrow C \hookrightarrow R' \hookrightarrow R'[x_n] \hookrightarrow S'$$

where R' is the integral closure of C in S'. This yields extensions:

(a) $R \hookrightarrow C \hookrightarrow R'$

(b) $R' \hookrightarrow R'[x_n] \hookrightarrow S$

By (3.13) S is quasi finite over R' in Q. Therefore by (3.16) $S_{P'} = R'_{P'}$ where $P' = Q \cap R'$.

We cannot directly apply (3.16) to the sequence (a). There are two obstacles: First R' may not be quasi finite over R in P' and second R' may not be a finite C-module. In order to avoid these difficulties we need to replace R' by an appropriate C-subalgebra. Write

$$R' = \varinjlim_{i \in I} R'_i$$

where $\{R'_i\}_{i \in I}$ is the set of finite C-subalgebras of R'. Put $P'_i = Q \cap R'_i = P' \cap R'_i$. For all $i \in I$ there is a natural morphism:

$$\varphi_i : (R'_i)_{P'_i} \longrightarrow R'_{P'} = S_{P'}.$$

<u>Claim</u>: There is a $j \in I$ so that for all $i \in I$ with $i \geq j$ the morphism φ_i is an isomorphism.

<u>Pf of Cl</u>: We know that S is a finitely generated R'-algebra and that $R'_{P'} = S_{P'}$. Hence there is an element $f \in R' - P'$ with $R'_f = S_f$. Write $S = R'[y_1, \ldots, y_t]$ and let $j \in I$ so that:

(α) $f \in R'_j$

(β) For some $k \in \mathbb{N}$: $f^k y_i \in R'_j$ for all $1 \leq i \leq t$.

Then $(R'_i)_f = S_f = R'_f$ for all $i \in I$ with $i \geq j$. In particular, $(R'_i)_{P'_i} = S_{P'_i}$ for all $i \in I$ with $i \geq j$. Since $S_{P'_i}$ is a quasi local ring, $R'_i - P'_i \subseteq R' - P'$, and $P'_i S_{P'} \subseteq P' S_{P'}$, we also have that $S_{P'_i} = S_{P'}$ for all $i \in I$ with $i \geq j$. This proves the claim.

In order to finish the proof of the lemma consider for all $i \in I$ with $i \geq j$ the morphisms: $R \hookrightarrow C = R[x_1, \ldots, x_{n-1}] \hookrightarrow R'_i \longrightarrow S$. With $f \in R' - P'$ as in the proof of the claim we have:

$$\begin{aligned}(S/PS)_Q = (S/PS)_{P'} &= (S \otimes_R k(P))_{P'_i} \\ &= (S_f \otimes_R k(P))_{P'_i} \quad \text{since } f \notin Q \\ &= ((R_i)_f \otimes_R k(P))_{P'_i} \quad \text{since } (R_i)_f = S_f \\ &= (R'_i/PR'_i)_{P'_i}\end{aligned}$$

Hence R'_i is quasi finite over R in P'_i and the induction hypothesis applies to the extensions $R \hookrightarrow C \hookrightarrow R_i'$. Therefore $R_P = (R_i')_P = (R_i')_{P_i'} = S_{P_i'}$. Let $f \in R_i' - P_i'$ with $(R_i')_f = S_f$ be as above. Since $(R_i')_{P_i'} = R_P$ write $f/1 = g/s \in R_P$ where $g, s \in R - P$. Then there is a $t \in R - P$ so that $f = g/s \in R'_{sgt}$. Therefore $S_{gst} = R'_{gst}$ and $S_P = R'_P = R_P$ as desired.

We have seen so far that in order to prove Zariski's Main Theorem it suffices to show Lemma (3.16). This will be done by showing three more Lemmas.

(3.18) <u>Lemma</u>: Let $R \hookrightarrow R[x] = S$ be a ring extension with R integrally closed in S. If S is quasi finite over R in $Q \in \operatorname{Spec}(S)$, then $R_P = S_P$ where $P = Q \cap R$.

<u>Proof</u>: (a) Consider the ring extension $R_P \longrightarrow (R[x])_P = R_P[x] = S_P$. Then:

(a1) R_P is integrally closed in S_P.

(a2) The fiber in P of the extension $R_P \longrightarrow S_P$ is the same as the fiber in P of the extension $R \hookrightarrow S$, namely: $S \otimes_R k(P) = S_P \otimes_{R_P} k(P)$.

Hence we may assume that R is a quasi local ring with maximal ideal P, $Q \subseteq S$ is a prime ideal which lies over P, and S is quasi finite over R in Q. We have to show that $S = R[x] = R$. Since R is integrally closed in S it suffices to show that x is integral over R. In the following let $k = k(P) = R/P$.

(b) <u>Claim</u>: The ring $S \otimes_R k = R[x] \otimes_R k = k[\bar{x}]$ is a finite k-algebra.

<u>Pf of</u> (b): By assumption the localization $(S/PS)_Q = (S \otimes_R k)_Q$ is a finite k-algebra. The ring $S \otimes_R k$ is either isomorphic to the polynomial ring $k[t]$ in one variable over k or $S \otimes_R k$ is a finite k-algebra. Since $(S \otimes_R k)_Q$ is finite over k, the ring $S \otimes_R k$ is finite over k.

(c) By (b) there is a monic polynomial $F(t) \in R[t]$ such that $F(x) \in PS$. Let $y = 1 + F(x)$ and consider the extensions

$$R \hookrightarrow R[y] \hookrightarrow R[x] = S.$$

Note that $R[x]$ is integral over $R[y]$ and it remains to show that y is integral over R.

(d) Claim: The fiber ring $R[y] \otimes_R k = k[\bar{y}]$ is a finite k-algebra.

Pf of (d): It suffices to show that $\dim k[\bar{y}] = 0$. Suppose not, then there are prime ideals $W_1, W_2 \in \operatorname{Spec}(R[y])$ such that:

(d1) $W_1 \subsetneq W_2$

(d2) $W_1 \cap R = P = W_2 \cap R$

Since S is integral over $R[y]$, there are prime ideals $Q_1, Q_2 \subseteq S$ such that

(d3) $Q_1 \subsetneq Q_2$

(d4) $Q_1 \cap R[y] = W_1$ and $Q_2 \cap R[y] = W_2$.

Thus $\dim S/PS \geq 1$, a contradiction since by (b) $S/PS = S \otimes_R k$ is a finite k-algebra. Hence $\dim k[\bar{y}] = 0$ and $k[\bar{y}]$ is finite over k.

(e) Claim: $\bar{y}$ is invertible in $k[\bar{y}]$.

Pf of (e): Suppose that $\bar{y}$ is not invertible in $k[\bar{y}]$. Then there is a prime ideal $W_0 \in \operatorname{Spec}(R[y])$ with

(e1) $y \in W_0$

(e2) $PR[y] \subseteq W_0$

Since $S = R[x]$ is integral over $R[y]$ there is a prime ideal $Q_0 \in \operatorname{Spec}(R[x])$ with:

(e3) $1 + F(x) \in Q_0$

(e4) $PR[x] \subseteq Q_0$.

Since $F(x) \in PR[x]$ we obtain $1 \in Q_0$, a contradiction.

(f) Claim: y is integral over R.

Pf of (f): By (d) and (e) $\bar{y}$ is integral over k and $\bar{y}$ is a unit in $k[\bar{y}]$. Let $n \in \mathbb{N}$ be minimal so that

$$\bar{y}^n + \bar{a}_{n-1}\bar{y}^{n-1} + \text{---} + \bar{a}_1\bar{y} + \bar{a}_0 = 0$$

where $\bar{a}_i \in k$ and not all $\bar{a}_i = 0$. Since $\bar{y}$ is a unit, $\bar{a}_0 \neq 0$. Let $a_i \in R$ so that $\bar{a}_i = a_i + P$ for all $0 \leq i \leq n-1$. Then

$$y^n + a_{n-1}y^{n-1} + \text{---} + a_1 y + a_0 \in PR[y].$$

Hence there is an $m \in \mathbb{N}$ and elements $p_i \in P$ so that

$$y^n + a_{n-1}y^{n-1} + \text{——} + a_1 y + a_0 = p_m y^m + p_{m-1}y^{m-1} + \text{——} + p_1 y + p_0 .$$

Let $t = \max(n,m)$ and set $a_j = 0$, or $p_k = 0$, accordingly, to obtain:

$$(a_0 - p_0) + (a_1 - p_1)y + \text{——} + (a_t - p_t)y^t = 0.$$

By assumption R is a quasi local ring with maximal ideal P and $a_0 - p_0 \notin P$. Therefore $a_0 - p_0$ is invertible in R yielding:

(f1) y is invertible in $R[y]$, i.e. $y^{-1} \in R[y]$

(f2) y^{-1} is integral over R.

Since R is integrally closed in S, we have that $y^{-1} \in R$. Since y^{-1} is invertible in S, $y^{-1} \notin Q$ and hence $y^{-1} \notin P$. Thus y^{-1} is invertible in R and $y \in R$.

(3.19) Lemma: Let R and S be domains, t a variable over R with $R[t] \subseteq S$ and S integral over $R[t]$. Suppose that $Q(R)(t) \subseteq Q(S)$ is a finite field extension. Then for all $Q \in \operatorname{Spec}(S)$ and $P = Q \cap R$ the ring $(S \otimes_R k(P))_Q$ has dimension ≥ 1 if Q is maximal lying over P. In particular, if S is of finite type over R, for all $Q \in \operatorname{Spec}(S)$ S is not quasi finite over R in Q.

Proof: Let $Q \subseteq S$ be a prime ideal, $P = Q \cap R$, and suppose Q is maximal among the prime ideals lying over P. We want to show that Q is not minimal over P.

Case 1: R is normal ($\triangleq$ integrally closed in $Q(R)$)

Then the polynomial ring $R[t]$ is also a normal domain (see, for example: Bourbaki: Commutative Algebra). We set $W = Q \cap R[t]$ and claim that W is maximal among the prime ideals of $R[t]$ which lie over P. Suppose that there is a prime ideal $W' \subseteq R[t]$ with $W \subseteq W'$ and $W' \cap R = P$. By Going-up there is a prime ideal $Q' \in \operatorname{Spec}(S)$ with $Q \subseteq Q'$ and $Q' \cap R[t] = W'$. Since $Q' \cap R = W' \cap R = P$ it follows that $Q = Q'$ and hence $W = W'$!

Let $\overline{W}$ denote the image of W in $R[t] \otimes_R k(P) = k(P)[t]$. Then $\overline{W}$ is maximal in $k(P)[t]$, in particular, $\overline{W} \neq 0$. The zero ideal of $k(P)[t]$ corresponds to a prime ideal $W_0 = PR[t] \subsetneq W$. By Going-down there is a prime ideal

$Q_0 \subseteq S$ with $Q_0 \subseteq Q$ and $Q_0 \cap R[t] = W_0$. Hence Q is not minimal over P.

<u>Case</u> 2: R an arbitrary domain.

Let $R' \subseteq Q(R)$ and $S' \subseteq Q(S)$ denote the integral closures of R and S, respectively.
There is the following commutative diagram of ring extensions:

$$\begin{array}{ccccc} R' & \hookrightarrow & R'[t] & \longrightarrow & S' \\ \uparrow & & \uparrow & & \uparrow \text{int.} \\ R & \hookrightarrow & R[t] & \xrightarrow{\text{int.}} & S \end{array}$$

S' is integral over $R[t]$ and $R'[t]$. Let $Q' \subseteq S'$ be a prime ideal with $Q' \cap S = Q$, then $P' = Q' \cap R'$ is a prime ideal with $P' \cap R = P$. Since Q is maximal over P, Q' is maximal over P'. By case 1 there is a prime ideal $W' \subsetneq Q'$ in S' with $W' \cap R' = P'$. Then $W' \cap S = W \subsetneq Q$ and $W \cap P = P$. Q is not minimal over P.

If S is quasi finite over R in some $Q \in \mathrm{Spec}(S)$, then $(S/PS)_Q$ is finite over $k(P)$ where $P = R \cap Q$. By (3.6) Q is maximal and minimal in the set of prime ideals lying over P.

(3.20) <u>Remark</u>: Since the Going-down Theorem for $R[t] \hookrightarrow S$ with R integrally closed also holds for infinite algebraic extensions $Q(R)(t) \subseteq Q(S)$, the lemma is true without any finiteness restrictions on the field extension $Q(R)(t) \subseteq Q(S)$.

(3.21) <u>Lemma</u>: Let $R \subseteq R[x] \subseteq S$ be ring extensions such that S is integral over $R[x]$ and R is integrally closed in S. If there is a monic polynomial $F(t) \in R[t]$ so that $F(x) S \subseteq R[x]$, then $S = R[x]$.

<u>Proof</u>: Suppose that $F(t) \in R[t]$ is a monic polynomial with $F(x) S \subseteq R[x]$.
Let $s \in S$. Since $F(x)s \in R[x]$ there is a polynomial $G(t) \in R[t]$ with $F(x)s = G(x)$. Since F is monic there are polynomials $P(t), H(t) \in R[t]$ with $\deg P(t) < \deg F(t)$ and $G(t) = H(t)F(t) + P(t)$. Hence $G(x) = F(x)s = H(x)F(x) + P(x)$. Let $y = s - H(x)$. Then $yF(x) = P(x)$ where $\deg P(t) < \deg F(t)$. It suffices to show that $y \in R[x]$.

Set $S' = S_y$ and let R', x', y' denote the images of R, x, y in S_y. Then $y' F(x') = P'(x')$ and hence $F(x') = y'^{-1} P(x')$ in S'. Consider P and F as polynomials in $R'[t]$. Since F is monic and $\deg F > \deg P$, the last equation implies that x' is integral over the ring $R'[y'^{-1}] \subseteq S'$.

On the other hand the element $y \in S$ is integral over $R[x]$ and therefore y' is integral over $R'[x']$. Thus y' is integral over $R'[y'^{-1}]$. This yields an equation:

$$y'^{m} + a_{m-1} y'^{m-1} + \dots + a_0 = 0$$

where $a_i \in R'[y'^{-1}]$. Let $t \in \mathbb{N}$ be the maximum of the degrees of the elements a_i, considered as polynomial expressions in y'^{-1}. Multiplying by y'^{t} yields an equation:

$$y'^{m+t} + b_{m+t-1} y'^{m+t-1} + \dots + b_1 y' + b_0 = 0$$

where $b_i \in R'$. This shows that y' is integral over R'.

Let $L(t) \in R[t]$ be a monic polynomial with $L(y') = 0$ in $S' = S_y$. Then there is an integer $k \in \mathbb{N}$ so that $y^k L(y) = 0$ in S. Thus y is integral over R and since R is integrally closed in S: $y \in R$.

We are now ready to finish the proof of Zariski's Main Theorem:

<u>Proof of (3.16)</u>: By assumption there are given ring extensions $R \hookrightarrow R[x] \hookrightarrow S$ where

(i) S is finite over $R[x]$

(ii) R is integrally closed in S.

Let $Q \subseteq S$ be a prime ideal so that S is quasifinite over R in Q. We have to show that $S_P = R_P$ where $P = Q \cap R$.

Consider the conductor of S in $R[x]$:

$$I = \{\alpha \in R[x] \mid \alpha S \subseteq R[x]\}.$$

Obviously, I is an ideal in $R[x]$. We distinguish two cases:

<u>Case 1</u>: $I \nsubseteq Q$

In this case $R[x]_W = S_Q$ where $W = Q \cap R[x]$. Then $R[x]_W \otimes_R k(P) = S_Q \otimes_R k(P)$ and $R[x]$ is quasi finite over R in W. Since R is integrally closed in $R[x]$ by

Lemma 3.18 : $R_P = R[x]_P$. Thus $R_P = R[x]_W = S_Q$. Let $S = R[x, y_1, \dots, y_\ell]$ and $\varphi : R \longrightarrow S_Q$ the induced morphism. Since $\varphi_P : R_P \longrightarrow S_Q$ is an isomorphism there is an $f \in R-P$ and element $b, a_1, \dots, a_\ell \in R$ so that $\varphi(b) = fx$ and $\varphi(a_i) = fy_i$ for all $1 \leq i \leq \ell$. Thus $R_f = S_f$, in particular $R_P = S_P$.

<u>Case 2:</u> $I \subseteq Q$

Let $N \in \mathrm{Spec}(S)$ be a minimal prime ideal with $I \subset N \subseteq Q$ and set $M = N \cap R$. Consider the commutative diagram:

$$\begin{tikzcd}
R/M \arrow[r] \arrow[d] & S/N \arrow[d] \\
k(M) \arrow[r] & k(N)
\end{tikzcd}$$

where $k(M) = (R/M)_M$ and $k(N) = (S/N)_N$. Let $\overline{x}$ denote the image of x in $k(N)$.

<u>Claim:</u> $\overline{x}$ is transcendental over $k(M)$

<u>Pf of Cl:</u> Localizing at $R-M$ yields ring extensions: $R_M \hookrightarrow (R[x])_M = R_M[x] \hookrightarrow S_M$. By assumption S_M is finite over $R_M[x]$ and R_M is integrally closed in S_M. Hence in order to prove the claim we may assume that $R = R_M$ is a quasi local ring with maximal ideal M and residue field $k = R/M$. With $N_0 = N \cap R[x]$ we have a commutative diagram of ring morphisms

$$\begin{tikzcd}
k = R/M \arrow[r, "\alpha"] \arrow[d, "\varepsilon"'] & R[x]/N_0 \arrow[r, "\beta"] & S/N \\
k[t] \arrow[ur, "\varphi"'] & &
\end{tikzcd}$$

where t is a variable over k and φ is defined by $\varphi(t) = \overline{x}$. Note that φ is a surjective map. Suppose that $\overline{x} \in R[x]/N_0$ is algebraic over k. Then $\ker(\varphi) \neq 0$ and $R[x]/N_0$ is a finite extension field of k. In particular, N_0 is a maximal ideal of $R[x]$. Hence N is a maximal ideal of S and $k(N) = S/N$.

Note the following facts about the conductor ideal I:

(a) The conductor behaves well under localization. In particular, I_M is the conductor of S_M over $R_M[x]$.

(b) $IS = I$ since $IS \subseteq R[x]$ and $1 \in S$.

Let $F(t) \in R[t]$ be a monic polynomial with $F(x) \in N$. Since N is a minimal prime divisor of IS, $\mathrm{rad}(IS_N) = NS_N$. Hence there is an $r \in \mathbb{N}$ so that $F(x)^r \in IS_N$. Let $y \in S - N$ so that $yF(x)^r \in IS = I$ yielding $yF(x)^r S \subseteq R[x]$.

With $S' = R[x][yS]$ we again have that R is integrally closed in S' and S' is integral over $R[x]$. Since $F(x)^r S' \subseteq R[x]$ by (3.21) $S' = R[x]$ and $yS \subseteq R[x]$. But then $y \in I$ contradicting $y \in S - N$. This proves the claim.

Consider the extensions $\overline{R} = R/M \hookrightarrow \overline{R}[\overline{x}] = R[x]/N_0 \hookrightarrow \overline{S} = S/N$. Set $\overline{Q} = Q/N$ and $\overline{P} = P/M$. If $\overline{S}$ is quasi finite over R in Q, then $\overline{Q}$ is maximal and minimal in the set of prime ideals lying over $\overline{P}$. This contradicts Lemma 3.19 since $\overline{R}[\overline{x}]$ is (isomorphic to) the polynomial ring over $\overline{R}$. (Note that $\overline{R}$ is a domain)

(3.22) <u>Corollary</u>: Let R be a ring and S an R-algebra of finite type. The set

$$\{Q \in \mathrm{Spec}(S) \mid S \text{ is quasi finite over } R \text{ in } Q\}$$

is an open subset of $\mathrm{Spec}(S)$.

<u>Proof</u>: Let R' denote the integral closure of R in S and let $Q \in \mathrm{Spec}(S)$ with S quasi finite over R in Q. By ZMT there is an element $f \in R'$, $f \notin Q$, so that $R'_f = S_f$. Since S is an R-algebra of finite type there is a finite R-subalgebra $R_0 \subseteq R'$ with $f \in R_0$ and $(R_0)_f = S_f$. Since R_0 is (quasi) finite over R in every prime ideal of $\mathrm{Spec}(R_0)$, S_f is quasi finite over R in every prime ideal of $\mathrm{Spec}(S_f)$. Hence $D_f = \mathrm{Spec}(S) - V(f) \subseteq \{Q \in \mathrm{Spec}(S) \mid S \text{ is quasi finite over } R \text{ in } Q\}$.

§3: LOCAL STRUCTURE OF UNRAMIFIED AND ÉTALE MORPHISMS

(3.23) Definition: Let R be a ring, S an R-algebra, and $Q \subseteq S$ a prime ideal. S is called unramified (étale) over R in a neighborhood of Q if there is an element $f \in S-Q$ such that S_f is unramified (étale) over R.

Recall definition (2.10) of standard étale algebras:
Let R be a ring, $f, g \in R[x]$ polynomials with f monic. The R-algebra $S = (R[x]/(f))_g$ is called a standard étale algebra over R if f' is invertible in S.

(3.24) Definition: Let R be a ring and S an R-algebra. S is called of finite presentation over R if $S \cong R[x_1, \ldots, x_n]/I$ where $x_1, \ldots, x_n$ are variables over R and $I \subseteq R[x_1, \ldots, x_n]$ a finitely generated ideal.

(3.25) Theorem: Let R be a ring, S an R-algebra of finite presentation and $Q \subseteq S$ a prime ideal. Let $P = Q \cap R$ denote the contraction of Q to R.

(a) The following conditions are equivalent:
  (a1) S is étale over R in a neighborhood of Q
  (a2) There are elements $f \in S-Q$ and $h \in R-P$ such that S_f is a standard étale algebra over R_h.

(b) The following conditions are equivalent:
  (b1) S is unramified over R in a neighborhood of Q.
  (b2) There are elements $f \in S-Q$ and $h \in R-P$ and a standard étale algebra C over R_h so that S_f is a homomorphic image of C.

Moreover, a surjective R_h-algebra morphism $u: C \longrightarrow S_f$ can be chosen such that the induced morphism $u \otimes_R k(P): C \otimes_R k(P) \longrightarrow S_f \otimes_R k(P)$ is an isomorphism.

Proof: (2) $\Rightarrow$ (1): (a) By (2.9) a standard étale algebra is étale. Therefore S_f is étale over R_h and by (2.8) and (2.4) S_f is étale over R.
(b) Since C is étale over R_h by (2.8) and (2.4) C is étale over R. By (2.3) and (2.4) a homomorphic image of C is unramified over R.
(1) $\Rightarrow$ (2): The proof requires several reduction steps. The first three steps are common for (a) and (b). We prove (b) in step 4 and (a) in step 5.
Step 1: Reduction to the case where R is quasi local with maximal ideal P.
By assumption (1) there is an element $f \in S - Q$ so that S_f is étale (unramified) over R_P. In particular, S_P is étale (unramified) over R_P in a neighborhood of QS_P. Suppose we know that there is an element $f \in S_P - QS_P$ and a standard étale R_P-algebra C so that in case (a) $C \cong (S_P)_f$ and in case (b) $C/I \cong (S_P)_f$ for some ideal $I \subseteq C$. We may suppose $f \in S - Q$ and write $C = (R_P[x]/(u))_g$ where $g, u \in R[x]$ polynomials, u monic, and u' invertible in C. By clearing denominators we can find an element $h_0 \in R - P$ and a standard étale R_{h_0}-algebra

$$C_0 = \left(R_{h_0}[x]/(u)\right)_g$$

so that $C = (C_0)_P$. We want to show that there is an element $h_1 \in R - P$ so that $S_f \cong (C_0)_{h_1}$ in case (a) and $S_f \cong (C_0/I_0)_{h_1}$ in case (b) where I_0 is the preimage of I in C_0. We only treat case (a). Write $S = R[y_1, -, y_n]/J$ where $y_1, -, y_n$ are variables over R and $J \subseteq R[y_1, -, y_n]$ is a finitely generated ideal. Also assume that $f \in R[y_1, -, y_n]$. The isomorphism $S_f \cong C$ defines an R-algebra morphism $\varphi: R[y_1, -, y_n]_f \longrightarrow C = (C_0)_P$. Since J is finitely generated there is a $h_1 \in R \setminus P$ so that the induced R-algebra morphism $\varphi_1: R[y_1, -, y_n]_f \longrightarrow (C_0)_{h_1}$ factors through S_f, say $\psi: S_f \longrightarrow (C_0)_{h_1}$. Consider the composition of R-algebra morphisms $S_f \xrightarrow{\psi} (C_0)_{h_1} \xrightarrow{\lambda} (C_0)_P$ where λ is the natural map. Since $\lambda\psi$ is an isomorphism and all elements of $R - P$ are invertible in S_f, all elements of $R - P$ are invertible in $(C_0)_{h_1}$ and λ is an isomorphism. Hence $S_f \cong (C_0)_{h_1}$ and $(C_0)_{h_1}$ is a standard étale algebra over R_h with $h = h_0 h_1$.

Assume from now on that (R,P,k) is a quasi-local ring, S an R-algebra of finite presentation and $Q \subseteq S$ a prime ideal with $P = Q \cap R$.

<u>Step 2</u>: Reduction to the case where S is finite over R.

We only treat the unramified case, the étale case follows by the same argument. Let S be unramified over R in a neighborhood of Q and $f \in S - Q$ so that S_f is unramified over R. By (2.5) $S_f \otimes_R k(P)$ is unramified over $k(P)$. Thus by (3.1) and (3.2) $S_f \otimes_R k(P)$ and $S_Q \otimes_R k(P)$ are finite over $k(P)$. Therefore S is quasi finite over R in Q. Let R' denote the integral closure of R in S. By Zariski's Main Theorem there is an element $g \in R'$ with $g \notin Q$ and $R'_g \cong S_g$. Since S is of finite type over R, there is a finite R-subalgebra $S' \subseteq R'$ with $S'_g \cong S_g$. Replace S by S'. In the following we assume that (R,P,k) is a quasi local ring with maximal ideal P, S is a finite R-algebra and $Q \subseteq S$ is a prime ideal with $P = Q \cap R$. Under these assumptions we want to show that $(1) \Rightarrow (2)$. Note that we do not assume anymore that S is of finite presentation over R.

<u>Step 3</u>: Reduction to the case where S is generated over R by a single element.

Since $\bar{S} = S \otimes_R k$ is finite over k, $\bar{S}$ is an Artinian ring and $\bar{Q} = Q/PS$ is a maximal ideal of $\bar{S}$. $\bar{S}$ is unramified over k in a neighborhood of $\bar{Q}$, thus $\bar{S}_{\bar{Q}}$ is unramified over k and by (3.3) $\bar{S}_Q$ is a finite separable field extension of k. Write $\bar{S}_Q = k(\alpha)$ for some element $\alpha \in \bar{k}$ and note that $\bar{S} = k(\alpha) \times \prod_{i=1}^{t} S_i$ where the S_i are local Artinian rings. Let $\bar{x} = (\alpha, 0, \ldots, 0) \in \bar{S}$. Then $\bar{S}_Q = k(\alpha) = k[\bar{x}]$. Let $x \in S$ be a preimage of $\bar{x}$, set $T = R[x] \subseteq S$, and $W = Q \cap T$. Note that $x \notin Q$, but x is contained in any other maximal ideal of S which contains PS. Thus $x \notin W$ and QS_W is the only prime ideal of S_W which contains PS_W. Hence $S_W \otimes_R k \cong S_W/PS_W = \bar{S}_Q = k(\alpha)$. Since also $T_W/WT_W \cong k(\alpha)$ we have that $T_W/WT_W \cong S_W/WS_W$ and therefore $S_W = T_W + WS_W$. Since S_W is a finite T_W-module, by Nakayama $S_W = T_W$. Since S and T are finite R-modules there is an element $f \in T - W$ with $S_f = T_f$. Hence we may replace S by T_f.

Step 4: (b1) $\Rightarrow$ (b2)

Let $S = R[x]$ be finite over R. Set $\bar{S} = S \otimes_R k = k[\bar{x}]$ with $\dim_k \bar{S} = r < \infty$. By Nakayama the elements $1, x, \ldots, x^{r-1}$ generate the R-module S. Hence there is a monic polynomial $u \in R[t]$ with $\deg u = r$ and a surjective R-algebra morphism $\varphi : R[t]/(u) \longrightarrow S$ with $\varphi(t) = x$. The induced morphism $\bar{\varphi} : k[t]/(u) \xrightarrow{\cong} \bar{S}$ is an isomorphism. Let $Q' = \bar{\varphi}^{-1}(\bar{Q})$ be the contraction of $\bar{Q} = Q(S/PS)_P = Q\bar{S}$ in $k[t]/(u)$. By assumption S is unramified over R in a neighborhood of Q, hence $\bar{S}_{\bar{Q}} = \bar{S}_Q$ is a finite separable field extension of k. Since $k[t]/(u) = \prod_{i=1}^t A_i$ is a product of local Artinian rings with, say $(k[t]/(u))_{Q'} = A_1 = \bar{S}_Q$, we have that $u = vw$ with $v, w \in k[t]$, v irreducible in $k[t]$, $(v,w) = 1$, and $\bar{S}_Q = k[t]/(v)$. Hence $v \in Q'$, $w \notin Q'$, and $v' \notin Q'$ since $\bar{S}_Q$ is separable over k. In particular, $u' \notin Q'$. Moreover, since $k[t]/(u)$ is a product of local Artinian rings there is an element $h \in R[t]$ such that

$$(k[t]/(u))_{Q'} = (k[t]/(u))_h = \bar{S}_Q .$$

With $g = hu'$ and $f = \varphi(g)$ the ring S_f is a homomorphic image of the standard étale algebra $(R[t]/(u))_g$. Note that φ has been constructed so that $\bar{\varphi}_f = \varphi_f \otimes k$ is an isomorphism.

Step 5: (a1) $\Rightarrow$ (a2)

Let S be an R-algebra of finite presentation, $Q \subseteq S$ a prime ideal, and S étale over R in a neighborhood of Q. Set $P = Q \cap R$. Let $f \in S - Q$ be so that S_f is étale over R. Since S_f is again an R-algebra of finite presentation we may replace S_f by S. Then S is unramified over R and by (b) there is a standard étale R-algebra:

$$C = (R[t]/(q))_g$$

where $q \in R[t]$ is a monic polynomial with q' invertible in C and a surjective R-algebra morphism $u : C \longrightarrow S$. By (b) C and u may be chosen so that $\bar{u} = u \otimes k(P) : \bar{C} = C \otimes_R k(P) \longrightarrow \bar{S} = S \otimes_R k(P)$ is an isomorphism. Since S is of finite presentation, $I = \ker(u)$ is a finitely generated C-ideal.

Thus it suffices to show that $I_W = 0$ or, equivalently, by Nakayama that $(I/I^2)_W = 0$
Consider the exact sequence of R-modules: $(*)$ $0 \to I/I^2 \to C/I^2 \xrightarrow{\tilde{u}} S \to 0$
and the commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\cong"] \arrow[dr, "v"] & C/I \\ R \arrow[u] \arrow[r] & C/I^2 \arrow[u, "u"'] \end{tikzcd}$$

Since S is étale over R, there is an R-algebra morphism $v: S \to C/I^2$ which lifts the isomorphism $S \cong C/I$. Hence the sequence $(*)$ is split exact and we obtain a short exact sequence:

$$0 \to (I/I^2) \otimes_R k(P) \longrightarrow (C/I^2) \otimes_R k(P) \xrightarrow{\tilde{u} \otimes k(P)} S \otimes_R k(P) \to 0$$

Since $u \otimes k(P)$ is an isomorphism, $\tilde{u} \otimes k(P)$ is an isomorphism and $(I/I^2) \otimes_R k(P) \cong (I/I^2 + PI)_P = 0$. Thus $(I/I^2 + WI)_W = 0$ and by Nakayama $(I/I^2)_W = 0$. This finishes the proof.

(3.26) __Corollary__: Let R be a ring and S an R-algebra of finite presentation. If S is étale over R, then S is flat over R.

__Proof__: It suffices to show that S_Q is flat over R for every prime ideal $Q \subseteq S$. Let $P = Q \cap R$ denote the contraction of Q to R. By (3.25) there are elements $f \in S - Q$, $h \in R - P$ and a monic polynomial $q \in R[x]$ so that

$$S_f \cong \left(R_h[x]/(q) \right)_g$$

where $g \in R_h[x]$ and q' invertible in S_f. Thus S_f is flat over R.

(3.27) __Theorem__: Let R be a ring, S an R-algebra of finite presentation, and $Q \subseteq S$ a prime ideal. The following conditions are equivalent:

(a) S is étale over R in a neighborhood of Q.

(b) $(\Omega_{S/R})_Q = 0$ and S_Q is flat over R.

Proof: (a) $\Rightarrow$ (b): Let $f \in S-Q$ with S_f étale over R. Then S_f is unramified over R and by (2.13) $(\Omega_{S/R})_f = \Omega_{S_f/R} = 0$. By (3.26) S_f is flat over R.

(b) $\Rightarrow$ (a): Step 1: We may assume that R is a quasi local ring with maximal ideal $P = Q \cap R$.

Suppose we know that S_P is étale over R_P in a neighborhood of QS_P. Then there are elements $f \in S_P - QS_P$ and $h \in R_P - PR_P$ such that:

$$(S_P)_f \cong \left({}^{(R_P)_h[x]}/_{(u)} \right)_g$$

where $u, g \in R[x]$, u monic, and u' invertible in $(S_P)_f$. Since S is of finite presentation over R there are elements $\tilde{f} \in S-Q$, $\tilde{h} \in R-P$, so that

$$S_{\tilde{f}} \cong \left({}^{R_{\tilde{h}}[x]}/_{(\tilde{u})} \right)_{\tilde{g}}$$

where $\tilde{u}, \tilde{g} \in R_{\tilde{h}}[x]$, $\tilde{u}$ monic, and $\tilde{u}'$ invertible in $S_{\tilde{f}}$. Hence S is étale over R in a neighborhood of Q. Hence we may assume that R is a quasi local ring, $R = R_P$.

Step 2: S is unramified over R in a neighborhood of Q.

Since S is of finite presentation over R, the module of differentials $\Omega_{S/R}$ is a finitely generated S-module. Hence $\mathrm{Supp}(\Omega_{S/R}) = V(J)$ where $J \subseteq S$ is an ideal with $\mathrm{rad}\, J = \mathrm{rad}(\mathrm{ann}(\Omega_{S/R}))$. Since $(\Omega_{S/R})_Q = 0$, there is an element $t \in S-Q$ with $(\Omega_{S/R})_t = 0$. By (2.13) S_t is unramified over R.

We replace S by S_t and assume that (R, P, k) is a quasi local ring, S is an R-algebra of finite presentation and $Q \in \mathrm{Spec}(S)$ with $P = Q \cap R$. Furthermore we assume that S is unramified over R, $(\Omega_{S/R})_Q = 0$, and S_Q is flat over R. Localizing further, if necessary, we obtain by (3.25) that there is a standard étale R-algebra C and a surjective R-algebra morphism $u: C \to S$ with $\bar{u} = u \otimes k: \bar{C} = C \otimes_R k \to \bar{S} = S \otimes_R k$ an isomorphism. Let $I = \ker(u)$ and $W = u^{-1}(Q) \subseteq C$. Since S is of finite presentation over R the C-ideal I is finitely generated.

Step 3: $I_W = 0$

Consider the exact sequence of R-modules $0 \to I_W \to C_W \xrightarrow{u_W} S_Q \to 0$.

Since S_Q is flat over R, $\mathrm{Tor}_1^R(S_Q, k) = 0$, and the sequence

$$0 \longrightarrow I_W \otimes_R k \longrightarrow C_W \otimes_R k \xrightarrow{u_W \otimes k} S_Q \otimes_R k \longrightarrow 0$$

is exact. By choice of u, $u_W \otimes k$ is an isomorphism and $I_W \otimes_R k = 0$. Since I_W is finitely generated over C_W, by Nakayama $I_W = 0$. Therefore there is an element $q \in C - W$ with $I_q = 0$. S_q is étale over R.

(3.27) <u>Theorem</u>: Let R be a ring and S an R-algebra of finite presentation. Suppose that S is unramified (étale) over R in Q for every $Q \in \mathrm{Spec}(S)$. Then S is unramified (étale) over R.

<u>Proof</u>: For all $Q \in \mathrm{Spec}(S)$ there is an $f \in S - Q$ with S_f unramified (étale) over R. Thus $\mathrm{Spec}(S) = \bigcup_{i \in I} D_{f_i}$ where S_{f_i} is unramified (étale) over R. Since $\mathrm{Spec}(S)$ is quasi compact, $\mathrm{Spec}(S) = \bigcup_{i=1}^n S_{f_i}$ where $f_1 + \cdots + f_n = 1$.

Let C be an R-algebra, $I \subseteq C$ an ideal with $I^2 = 0$. Set $\overline{C} = C/I$.

The unramified case:

Suppose that there is given a commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\overline{u}"] \arrow[dr, shift left, "v"] \arrow[dr, shift right, "u"'] & \overline{C} \\ R \arrow[u, "\rho"] \arrow[r] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where u and v are liftings of $\overline{u}$. For all $1 \le i \le n$ there is a commutative diagram:

$$\begin{tikzcd} S_{f_i} \arrow[r, "\overline{u}_{f_i}"] \arrow[dr, shift left, "v_{f_i}"] \arrow[dr, shift right, "u_{f_i}"'] & \overline{C}_{f_i} \\ R \arrow[u, "\rho_{f_i}"] \arrow[r] & C_{f_i} \arrow[u, "\nu"'] \end{tikzcd}$$

where the R-algebra C_{f_i} is defined as follows: Let f_i also denote a preimage of $\overline{u}(f_i)$ in C. Since I is nilpotent the element $f_i + e$ is not contained in any prime ideal of C_{f_i}, thus $f_i + e$ is invertible in C_{f_i} and $C_{f_i} = C_{(f_i+e)}$.

Since S_{f_i} is unramified over R, for all $1 \le i \le n$ $u_{f_i} = v_{f_i}$. Hence, for all $s \in S$ there is an $m \in \mathbb{N}$ with $f_i^m(u(s) - v(s)) = 0$ for all $1 \le i \le n$. Hence

$(f_1, \dots, f_n) \subseteq \operatorname{rad}(\operatorname{ann}(u(s)-v(s)))$ and $u(s)=v(s)$ for all $s \in S$.

The étale case:

Consider a commutative diagram of ring morphisms:

$$\begin{tikzcd} S \arrow[r, "\bar{u}"] & \bar{C} \\ R \arrow[u, "\rho"] \arrow[r] & C \arrow[u, "\nu"'] \end{tikzcd}$$

Since S_{f_i} is étale over R, for all $1 \leq i \leq n$ there are R-algebra morphisms u_i so that the diagram:

$$\begin{tikzcd} S_{f_i} \arrow[r, "\bar{u}_{f_i}"] \arrow[dr, "u_i"'] & \bar{C}_{f_i} \\ R \arrow[u, "\rho_{f_i}"] \arrow[r] & C_{f_i} \arrow[u, "\nu"'] \end{tikzcd}$$

commutes. For all $1 \leq i, j \leq n$ u_i and u_j induce R-algebra morphism $u_{i f_j}, u_{j f_i} : S_{f_i f_j} \longrightarrow C_{f_i f_j}$ which lift $\bar{u}_{f_i f_j}$. Since $S_{f_i f_j}$ is unramified over R, for all $1 \leq i, j \leq n$: $u_{i f_j} = u_{j f_i}$.

In order to construct a morphism $u : S \longrightarrow C$ which lifts $\bar{u}$ let $x \in S$. Then there is an $r \in \mathbb{N}$ so that for all $1 \leq i, j \leq n$ there are elements $c_i, c_j \in C$ with

$$u_i(x) = c_i / f_i^r \text{ in } C_{f_i} \quad \text{and} \quad c_i/f_i^r = c_j/f_j^r \text{ in } C_{f_i f_j}.$$

Hence there is an $m \in \mathbb{N}$ so that for all $1 \leq i, j \leq n$:

$$f_j^m c_j = c_i f_j^{r+m}/f_i^r \quad \text{in } C_{f_i}.$$

Replacing $m+r$ by t and $c_i f_i^m$ by c_i' we have $c_j' = c_i' f_j^t / f_i^t$ in C_i for all $1 \leq i, j \leq n$.

Let $a_i \in S$ be such that $\sum_{i=1}^n a_i f_i^t = 1$. Define $u : S \longrightarrow C$ by $u(x) = \sum_{j=1}^n a_j c_j'$. Since $a_j c_j' = a_j c_i' f_j^t / f_i^t \in C_{f_i}$ we have that

$$u_{f_i}(x) = \sum_{j=1}^n a_j c_j' = \sum_{j=1}^n a_j c_i' f_j^t / f_i^t = c_i / f_i^r$$

and $u_{f_i} = u_i$. Verify that u is a well-defined R-algebra morphism which lifts u.

(3.28) <u>Corollary</u>: Let R be a ring and S an R-algebra of finite presentation. The following conditions are equivalent:

(a) S is étale over R.

(b) S is unramified and flat over R.

(c) $\Omega_{S/R} = 0$ and S is flat over R.

APPENDIX: ALGEBRAS OF FINITE PRESENTATION

Let R be a ring and S an R-algebra. S is called an R-algebra of finite presentation if S is isomorphic to an R-algebra $R[x_1,\dots,x_n]/I$ where $x_1,\dots,x_n$ are variables over R and I is a finitely generated ideal of $R[x_1,\dots,x_n]$.

(3.29) Proposition: Let R be a ring

(a) If $R \to S \to T$ are ring morphisms with S an R-algebra of finite presentation and T an S-algebra of finite presentation, then T is an R-algebra of finite presentation.

(b) If T is an R-algebra and S an R-algebra of finite presentation, then $S \otimes_R T$ is a T-algebra of finite presentation.

(c) If $f \in R$, then R_f is an R-algebra of finite presentation.

Proof: (a) and (b) are obvious. (c) follows since $R_f \cong R[x]/(1-xf)$ where x is a variable over R.

(3.30) Proposition: (a) Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be morphisms of rings with T an R-algebra of finite presentation and S an R-algebra of finite type. Then T is an S-algebra of finite presentation.

(b) Let $\varphi: R \to S$ be a surjective morphism of rings, S an R-algebra (via φ) of finite presentation. Then $\ker(\varphi)$ is a finitely generated ideal.

(c) Let S be an R-algebra of finite presentation, $z_1,\dots,z_m$ variables over R and $\delta: R[z_1,\dots,z_m] \to S$ a surjective R-algebra morphism. Then $\ker(\delta)$ is a finitely generated ideal.

Proof: (a) Consider the following commutative diagram of ring morphisms:

$$\begin{tikzcd} S\otimes_R S \arrow[r, "\alpha"] \arrow[d, "1\otimes\psi"'] & S \arrow[d, "\psi"] \\ S\otimes_R T \arrow[r, "\beta"] & T \end{tikzcd}$$

where $\alpha(s_1\otimes s_2)=s_1s_2$ and $\beta(s\otimes t)=\psi(s)t$. Notice that T is a push out and therefore $T\cong S\otimes_{S\otimes_R S}(S\otimes_R T)$.

We know that $\ker(\alpha)$ is generated by $\{1\otimes s - s\otimes 1 \mid s\in S\}$. Since S is an R-algebra of finite type $\ker(\alpha)$ is a finitely generated ideal of $S\otimes_R S$. Hence S is of finite presentation over $S\otimes_R S$. By (3.29) T is an algebra of finite presentation over $S\otimes_R T$.

The morphism ψ factors into: $S\xrightarrow{\gamma} S\otimes_R T\xrightarrow{\beta} T$ where $\gamma(s)=s\otimes 1$. Since T is of finite presentation over R, $S\otimes_R T$ is of finite representation over S (by (3.29)). Hence by by (3.30)(a) T is of finite presentation over S.

(b) Since S is of finite presentation, there are variables $x_1,\ldots,x_n$ and a surjective R-algebra morphism $\phi: R[x_1,\ldots,x_n]\longrightarrow S$ with $\phi|_R=\varphi$ so that $\ker(\phi)$ is a finitely generated ideal. Consider the commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "id"'] & R[x_1,\ldots,x_n] \arrow[d, "\phi"] \\ R \arrow[r, "\varphi"] & S \end{tikzcd}$$

Since φ is surjective there is an R-algebra morphism $\tau: R[x_1,\ldots,x_n]\longrightarrow R$ with $\varphi\tau=\phi$. Then $\tau(\ker(\phi))=\ker(\varphi)$ and $\ker(\varphi)$ is finitely generated.

(c) Consider the sequence of ring morphisms: $R\longrightarrow R[z_1,\ldots,z_m]\xrightarrow{\delta} S$. By (a) S is an $R[z_1,\ldots,z_m]$-algebra of finite presentation and by (b) $\ker(\delta)$ is a finitely generated ideal.

(3.31) <u>Definition</u>: Let R be a ring and S an R-algebra. S is called <u>essentially of finite presentation</u> over R if there are variables $x_1,\ldots,x_n$ so that $S\cong W^{-1}(R[x_1,\ldots,x_n]/I)$ where $I\subseteq R[x_1,\ldots,x_n]$ is a finitely generated ideal and $W\subseteq R[x_1,\ldots,x_n]$ a multiplicatively closed subset.

(3.31) Proposition: (a) Let $R \to S \to T$ be ring morphisms with S an R-algebra essentially of finite presentation and T an S-algebra essentially of finite presentation. Then T is an R-algebra essentially of finite presentation.

(b) If T is an R-algebra and S an R-algebra essentially of finite presentation, then $S \otimes_R T$ is a T-algebra essentially of finite presentation.

(c) Let $R \to S \to T$ be morphisms of rings with T an R-algebra essentially of finite presentation and S an R-algebra essentially of finite type. Then T is an S-algebra essentially of finite presentation.

(d) Let R and S be quasi local rings and $\varphi: R \to S$ a surjective local morphism of rings. If S is essentially of finite presentation over R, $\ker(\varphi)$ is finitely generated.

Proof: similar to (3.29) and (3.30)

Also see: B. Iversen: Generic local structure of the morphisms in Commutative algebra
Springer Lecture Notes in Mathematics, 310