CHAPTER VI: ARTIN APPROXIMATION

§1: WEIERSTRASS PREPARATION THEOREM

(6.1) Theorem: (Weierstraß preparation theorem for polynomial rings) Let K be a field, $R = K[x_1, \_, x_n]$ the polynomial ring over K, and $f \in R$ a polynomial with total degree $\deg(f) > 0$. Then there are integers $e_i \in \mathbb{N}$, $1 \le i \le n-1$, such that by substituting $t_i = x_i - x_n^{e_i}$ for $1 \le i \le n-1$ we obtain:

$$\begin{aligned} f(x_1, \_, x_n) &= f(t_1 + x_n^{e_1}, \_\_, t_{n-1} + x_n^{e_{n-1}}, x_n) \\ &= \varepsilon (x_n^k + a_{k-1} x_n^{k-1} + \_\_ + a_0) \end{aligned}$$

where $\varepsilon \in K$ and $a_i \in K[t_1, \_, t_{n-1}]$ for $0 \le i \le k-1$.

Proof: Write
$$f = \sum_{j=1}^{t} \Big(\sum_{i_1 + \dots + i_n = j} a_{(i)} x_1^{i_1} \dots x_n^{i_n} \Big)$$
where $a_{(i)} \in K$ and $(i) = (i_1, \_, i_n) \in \mathbb{N}^n$. With $q = t+1$ set $e_i = q^i$ for $1 \le i \le n-1$ and substitute $t_i + x_n^{e_i}$ for x_i $(1 \le i \le n-1)$ in f. For each $(i) \in \mathbb{N}^n$ with $a_{(i)} \neq 0$ we obtain:

$$g_{(i)} = a_{(i)} x_1^{i_1} \dots x_n^{i_n} = a_{(i)} (t_1 + x_n^{e_1})(t_2 + x_n^{e_2}) \_\_ (t_{n-1} + x_n^{e_{n-1}}) x_n^{i_n}.$$

$g_{(i)}$ contains the unique unmixed term $a_{(i)} x_n^{d_{(i)}}$ where
$$d_{(i)} = i_n + i_1 q + \dots + i_{n-1} q^{n-1}.$$

By the choice of q, if $(i), (j) \in \mathbb{N}^n$ with $|i|, |j| \le t$ and $(i) \neq (j)$ then $d_{(i)} \neq d_{(j)}$. In particular, the unmixed terms (in x_n) of $g_{(i)}$ and $g_{(j)}$ do not cancel each other. Thus for $(\ell) \in \mathbb{N}^n$ with
$$d_{(\ell)} = \max \{ d_{(i)} \mid a_{(i)} \neq 0 \}$$
we obtain
$$f = a_{(\ell)} (x_n^{d_{(\ell)}} + b_{d(\ell)-1} x_n^{d_{(\ell)}-1} + \dots + b_1 x_n + b_0)$$
where $b_j \in K[t_1, \_, t_{n-1}]$ for all $0 \le j \le d_{(\ell)} - 1$.

(6.2) Remark: Let K be a field, $x_1, \dots, x_n, t_1, \dots, t_{n-1}$ variables over K, and $e_i \in \mathbb{N}$ for $1 \le i \le n-1$. The K-algebra morphism

$$\varphi : K[x_1, \dots, x_n] \longrightarrow K[t_1, \dots, t_{n-1}, x_n]$$

defined by $\varphi(x_i) = t_i + x_n^{e_i}$ if $1 \le i \le n-1$ and $\varphi(x_n) = x_n$ is an isomorphism of the polynomial rings. By identifying $t_i = x_i$ for $1 \le i \le n-1$, φ defines a K-automorphism of the polynomial ring. φ is called a Weierstrass automorphism of $K[x_1, \dots, x_n]$. Theorem (6.1) says that after a suitable Weierstrass automorphism every nonzero polynomial $f \in K[x_1, \dots, x_n]$ is a product of a monic polynomial in x_n and an invertible element in K.

We want to show a similar theorem for the power series ring in finitely many variables over a field. We start with a definition and a proposition:

(6.3) Definition: Let K be a field, $x_1, \dots, x_n$ variables over K and $f \in (x_1, \dots, x_n) K[\![x_1, \dots, x_n]\!]$ a nonzero power series. f is called regular in the variable x_j if $f(0, \dots, x_j, \dots, 0) \neq 0$, that is, if

$$f = \sum_{|i| \ge 1} a_{(i)} x_1^{i_1} \cdots x_n^{i_n}$$

then there is an $h \in \mathbb{N} - (0)$ with $a_{(0, \dots, h, \dots, 0)} \neq 0$ (h at the j^{th} place).

(6.4) Remark: Note that $f = \sum_{|i| \ge 0} a_{(i)} x_1^{i_1} \cdots x_n^{i_n} \in K[\![x_1, \dots, x_n]\!]$ is invertible if and only if $a_{(0)} \neq 0$.

(6.5) Proposition: Let K be a field and $f \in (x_1, \dots, x_n) K[\![x_1, \dots, x_n]\!]$ a nonzero power series. There are integers $e_i \in \mathbb{N}$, $1 \le i \le n-1$, so that after substituting $t_i = x_i - x_n^{e_i}$ for $1 \le i \le n-1$ the power series

$$f(x_1, \dots, x_n) = f(t_1 + x_n^{e_1}, \dots, t_{n-1} + x_n^{e_{n-1}}, x_n)$$

is regular in x_n as a power series in $t_1, \dots, t_{n-1}, x_n$.

Proof: Write
$$f = \sum_{(i)\in\mathbb{N}^n} a_{(i)} x_1^{i_1} \cdots x_n^{i_n}$$
and let $j_{n-1} \in \mathbb{N}$ be minimal with $a_{(i_1,\dots,i_{n-2},j_{n-1},i_n)} \neq 0$, $j_{n-2}\in\mathbb{N}$ minimal with $a_{(i_1,\dots,i_{n-3},j_{n-2},j_{n-1},i_n)}\neq 0$ and so on. Finally let $j_n \in \mathbb{N}$ so that
$$j_n = \min\{ i_n \mid a_{(j_1,\dots,j_{n-1},i_n)} \neq 0\}.$$
The n-tuple $(j) = (j_1,\dots,j_n)\in\mathbb{N}^n$ has the following properties:

(a) $a_{(j_1,\dots,j_n)} \neq 0$

(b) For all $(i)\in\mathbb{N}^n$ with $a_{(i)}\neq 0$ and $(i)\neq(j)$ either $i_k = j_k$ for all $1\le k\le n-1$ and $i_n > j_n$ or with $k\in\{1,\dots,n-1\}$ maximal with $i_k\neq j_k$ it holds that $i_k > j_k$.

Let $q = \max\{j_1,\dots,j_n\}+1$ and set for all $(i)\in\mathbb{N}^n$:
$$d_{(i)} = i_n + i_1 q + \dots + i_{n-1} q^{n-1}.$$
We claim that for all $(i)\in\mathbb{N}^n$ with $a_{(i)}\neq 0$ and $(i)\neq(j)$ it holds that $d_{(i)}\neq d_{(j)}$. If $(i) = (j_1,\dots,j_{n-1},i_n)$ then $i_n > j_n$ and $d_{(i)}\neq d_{(j)}$. Suppose that there is a $k\in\{1,\dots,n-1\}$ maximal with $i_k\neq j_k$. Then $i_k > j_k$. Assume that
$$d_{(j)} = j_n + j_1 q + \dots + j_{n-1} q^{n-1} = i_n + i_1 q + \dots + i_{n-1} q^{n-1} = d_{(i)}.$$
Then $$j_n + j_1 q + \dots + j_k q^k = i_n + i_1 q + \dots + i_k q^k$$
and since $i_k > j_k$:
$$j_n + j_1 q + \dots + j_{k-1} q^{k-1} > q^k.$$
On the other hand, since $j_\ell \le q-1$ for all $1\le \ell\le n$:
$$j_n + j_1 q + \dots + j_{k-1} q^{k-1} \le (q-1)(1+q+\dots+q^{k-1})$$
$$= q^k - 1 < q^k,$$
a contradiction. Thus $d_{(i)}\neq d_{(j)}$ for all $(i)\in\mathbb{N}^n$ with $a_{(i)}\neq 0$ and $(i)\neq(j)$.

For all $(i)\in\mathbb{N}^n$ the polynomial in $t_1,\dots,t_{n-1},x_n$
$$g_{(i)} = a_{(i)} x_1^{i_1}\cdots x_n^{i_n} = a_{(i)} (t_1 + x_n^q)^{i_1} \cdots (t_{n-1} + x_n^{q^{n-1}})^{i_{n-1}} x_n^{i_n}$$
has a single unmixed term (in x_n): $a_{(i)} x_n^{d(i)}$. Therefore for all $(i)\in\mathbb{N}^n$ $a_{(i)} x_n^{d(i)}$ does not cancel $a_{(j)} x_n^{d(j)}$. Thus $f(t_1 + x_n^q, \dots, t_{n-1} + x_n^{q^{n-1}}, x_n)$ is regular in x_n (as a power series in $t_1,\dots,t_{n-1},x_n$).

(6.6) Theorem: Let K be a field and $f(x) \in (x_1, \dots, x_n) K[\![x_1, \dots, x_n]\!]$ a nonzero power series. Suppose that
$$f = \sum_{|i| \geq 1} a_{(i)} x_1^{i_1} \dots x_n^{i_n},$$
$a_{(i)} \in K$, is regular in x_n and that $s \in \mathbb{N}$ is minimal with $a_{(0,\dots,0,s)} \neq 0$. Then for all $g \in K[\![x_1, \dots, x_n]\!]$ there are unique power series $u \in K[\![x_1, \dots, x_n]\!]$ and $r_i \in K[\![x_1, \dots, x_{n-1}]\!]$, $0 \leq i \leq s-1$, so that
$$g = uf + \sum_{i=0}^{s-1} r_i x_n^i.$$

Proof: Consider K-linear maps:
$$R, H : K[\![x_1, \dots, x_n]\!] \longrightarrow K[\![x_1, \dots, x_n]\!]$$
defined by: for $p = \sum_{(i) \in \mathbb{N}^n} b_{(i)} x_1^{i_1} \dots x_n^{i_n}$ set:
$$R(p) = \sum_{\substack{(i) \in \mathbb{N}^n \\ i_n < s}} b_{(i)} x_1^{i_1} \dots x_n^{i_n} \quad \text{and} \quad H(p) = \frac{1}{x_n^s} \sum_{\substack{(i) \in \mathbb{N}^n \\ i_n \geq s}} b_{(i)} x_1^{i_1} \dots x_n^{i_n}.$$
Obviously, $p = R(p) + x_n^s H(p)$ where $H(p) \in K[\![x_1, \dots, x_n]\!]$ and $R(p) \in K[\![x_1, \dots, x_{n-1}]\!][x_n]$ a polynomial in x_n of degree $\deg_{x_n} R(p) < s$. Since $s \in \mathbb{N}$ is minimal with $a_{(0,\dots,0,s)} \neq 0$ we have that $R(f) \in (x_1, \dots, x_{n-1}) K[\![x_1, \dots, x_n]\!]$ and $H(f)$ is invertible in $K[\![x_1, \dots, x_n]\!]$.

Claim: Let $g, u \in K[\![x_1, \dots, x_n]\!]$. Then
$$H(g) = H(uf) \iff \exists\ r_i \in K[\![x_1, \dots, x_{n-1}]\!],\ 0 \leq i \leq s-1, \text{ with } g = uf + \sum_{i=0}^{s-1} r_i x_n^i.$$
Pf of Cl: "$\Rightarrow$" Let $g, u \in K[\![x_1, \dots, x_n]\!]$ with $H(g) = H(uf)$. Then $H(g - uf) = 0$ and $g - uf \in K[\![x_1, \dots, x_{n-1}]\!][x_n]$ is a polynomial in x_n of degree $\deg_{x_n}(g - uf) < s$. Hence there are $r_i \in K[\![x_1, \dots, x_{n-1}]\!]$, $0 \leq i \leq s-1$, so that $g - uf = \sum_{i=0}^{s-1} r_i x_n^i$.
"$\Leftarrow$": Conversely, if $g - uf = \sum_{i=1}^{s-1} r_i x_n^i$ with $r_i \in K[\![x_1, \dots, x_{n-1}]\!]$ for all $0 \leq i \leq s-1$, then $H(g - uf) = 0$ and $H(g) = H(uf)$, since H is K-linear.

In order to prove the theorem for some $g \in K[\![x_1, \dots, x_n]\!]$ by the claim we need to find a power series $u \in K[\![x_1, \dots, x_n]\!]$ so that $H(g) = H(uf)$.

First note that for all $u \in K[\![x_1, \dots, x_n]\!]$:
$$uf = u(R(f) + x_n^s H(f)) = uR(f) + u x_n^s H(f)$$

and therefore $\quad H(uf) = H(uR(f)) + uH(f)$.

By the claim we have to show that

$$H(g) = H(uR(f)) + uH(f)$$

for some $u \in K[\![x_1, \dots, x_n]\!]$. By assumption on f $H(f)$ is invertible in $K[\![x_1, \dots, x_n]\!]$ and $R(f) \in (x_1, \dots, x_{n-1})K[\![x_1, \dots, x_n]\!]$. Hence by setting $v = uH(f)$ we obtain:

$$H(uf) = H(vR(f)H(f)^{-1}) + v.$$

Setting $M = -R(f)H(f)^{-1}$ it suffices to find a power series $v \in K[\![x_1, \dots, x_n]\!]$ so that:

$$H(g) = -H(vM) + v.$$

Let N be the composition of the K-linear maps:

$$\underbrace{K[\![x_1, \dots, x_n]\!] \xrightarrow{\times M} K[\![x_1, \dots, x_n]\!]}_{N} \xrightarrow{H} K[\![x_1, \dots, x_n]\!].$$

Then for all $p \in K[\![x_1, \dots, x_n]\!]$ $\quad N(p) = H(Mp) = -H(R(f)H(f)^{-1}p)$. If $p \in (x_1, \dots, x_{n-1})^j K[\![x_1, \dots, x_n]\!]$ then $N(p) \in (x_1, \dots, x_{n-1})^{j+1} K[\![x_1, \dots, x_n]\!]$ since $R(f) \in (x_1, \dots, x_{n-1}) K[\![x_1, \dots, x_n]\!]$.

Suppose there is a $v \in K[\![x_1, \dots, x_n]\!]$ with $v = H(g) + N(v)$. Then

$$\begin{aligned} v &= H(g) + N(v) \\ &= H(g) + N(H(g) + N(v)) \\ &= H(g) + N(H(g)) + N^2(H(g) + N(v)) \\ &= H(g) + N(H(g)) + N^2(H(g)) + N^3(v) \end{aligned}$$

Since $N^k(p) \in (x_1, \dots, x_{n-1})^k K[\![x_1, \dots, x_n]\!]$ for all $p \in K[\![x_1, \dots, x_n]\!]$ we may define:

$$v = \sum_{k=0}^{\infty} N^k(H(g)) \in K[\![x_1, \dots, x_n]\!]$$

where N^0 denotes the identity map. In order to show that $v = H(g) + N(v)$ let $q \in \mathbb{N}$ and write $v = \sum_{k=0}^{q} N^k(H(g)) + w_q$ where $w_q \in (x_1, \dots, x_{n-1})^{q+1} K[\![x_1, \dots, x_n]\!]$. Then:

$$\begin{aligned} v - H(g) - N(v) &= \sum_{k=0}^{q} N^k(H(g)) + w_q - H(g) - \sum_{k=0}^{q-1} N^{k+1}(H(g)) - N(w_q) \\ &= w_q - N^{q+1}(H(g)) - N(w_q) \\ &\in (x_1, \dots, x_{n-1})^{q+1} K[\![x_1, \dots, x_n]\!]. \end{aligned}$$

Hence with this choice of v : $v = H(g) + N(v)$ and we have shown that $g = uf + \sum_{i=0}^{s-1} r_i x_n^i$ for some $u \in K[\![x_1,\ldots,x_n]\!]$ and $r_i \in K[\![x_1,\ldots,x_{n-1}]\!]$ for all $0 \le i \le s-1$.

The proof of the existence shows that $v = \sum_{k=0}^{\infty} N^k(H(g))$ is uniquely determined by f and g. Thus u and the elements $r_i \in K[\![x_1,\ldots,x_{n-1}]\!]$ are also uniquely determined.

(6.7) <u>Corollary</u>: (Weierstrass preparation theorem for power series rings) Let $f = \sum_{(i)\in\mathbb{N}^n} a_{(i)} x_1^{i_1}\cdots x_n^{i_n} \in (x_1,\ldots,x_n)K[\![x_1,\ldots,x_n]\!]$ be regular in x_n and $n \in \mathbb{N}$ minimal with $a_{(0,\ldots,0,s)} \neq 0$. Then there are a unit $\varepsilon \in K[\![x_1,\ldots,x_n]\!]$ and power series $r_i \in K[\![x_1,\ldots,x_{n-1}]\!]$ for $0 \le i \le s-1$ such that

$$f = \varepsilon\left(x_n^s + r_{s-1}x_n^{s-1} + \cdots + r_0\right).$$

Moreover, ε and r_i, $0 \le i \le s-1$, are uniquely determined by f.

<u>Proof</u>: By (6.6) there are power series $u \in K[\![x_1,\ldots,x_n]\!]$ and $r_i \in K[\![x_1,\ldots,x_{n-1}]\!]$ so that:

$$-x_n^s = uf + \sum_{i=0}^{s-1} r_i x_n^i \qquad \text{or} \qquad -uf = x_n^s + \sum_{i=0}^{s-1} r_i x_n^i .$$

It remains to show that u is invertible, or equivalently, $u(0,\ldots,0) \neq 0$. Obviously, $f(0,\ldots,0,x_n) = a_{(0,\ldots,0,s)} x_n^s + x_n^{s+1} h$ where $h \in K[\![x_n]\!]$. Hence:

$$-u(0,\ldots,0,x_n)\left(a_{(0,\ldots,0,s)} x_n^s + x_n^{s+1} h\right) = x_n^s + r_{s-1}(0)\, x_n^{s-1} + \cdots + r_0(0).$$

Thus $u(0,\ldots,0,x_n)$ is a unit in $K[\![x_n]\!]$, or equivalently, $u(0,\ldots,0) \neq 0$.

Uniqueness follows by (6.6).

§2: ARTIN APPROXIMATION

(6.8) Theorem: (M. Artin) Let K be a field, $S = K[x_1, -, x_n]$ the polynomial ring over K and $R = (K[x_1,-,x_n]_{(x_1,-,x_n)})^h$ the Henselization of the localized polynomial ring, $m = (x_1, -, x_n)R$ the maximal ideal of R. Let $y = (y_1, -, y_N)$ be variables over R and $f = (f_1, -, f_m)$ an ideal of $R[y_1, -, y_N]$. If $\hat{a} = (\hat{a}_1, -, \hat{a}_N) \in \hat{R}^N$ is a solution of the system of equations $f = 0$, then for every $c \in \mathbb{N}$ there is an element $a = (a_1, -, a_N) \in R^N$ with $f(a) = 0$ and $a_i \equiv \hat{a}_i \bmod m^c\hat{R}$ for all $1 \leq i \leq N$.

The remainder of this section is devoted to the proof of theorem (6.6).

(6.9) Definition: Let (R, m) be a local Noetherian ring. R has the approximation property if for all $f = (f_1, -, f_m) \subseteq R[y_1, -, y_N]$, where $y_1, -, y_N$ are variables over R, it holds that the system $f = 0$ has a solution in R if and only if $f = 0$ has a solution in $\hat{R}$.

(6.10) Remark: (a) If the ring (R, m) has the approximation property, then R is Henselian.

(b) Let (R, m) be a ring with approximation property, $f = (f_1, -, f_m) \subseteq R[y_1, -, y_N]$ an ideal in the polynomial ring over R and $\hat{a} = (\hat{a}_1, -, \hat{a}_N) \in \hat{R}^N$ with $f(a) = 0$. Then for every $c \in \mathbb{N}$ there is an element $a = (a_1, -, a_N) \in R^N$ with $f(a) = 0$ and $a_i \equiv \hat{a}_i \bmod m^c\hat{R}$ for all $1 \leq i \leq N$.

Proof: (a) If (R, m) is a local Noetherian ring and R is not Henselian, then there is an étale neighborhood $S = (R[x]/(f))_n$ of R with $S \neq R$ and $f \in R[x]$ a monic polynomial which has no root in R. Since $\hat{R}$ is Henselian, f has a root in $\hat{R}$.

(b) Let $f = (f_1, -, f_m) \subseteq R[y_1, -, y_N]$ be an ideal and $\hat{a} = (\hat{a}_1, -, \hat{a}_N) \in \hat{R}^N$

with $f(\hat{a})=0$. For any $c\in\mathbb{N}$ consider a system of generators of $m^c=(g_1,\dots,g_r)$ and for all $1\le i\le N$ let $a_{i0}\in R$ with $\hat{a}_i-a_{i0}\in m^c\hat{R}$. Let z_{ij}, $1\le i\le N$; $1\le j\le r$, be additional variables. Consider the system of equations over R:

$$\begin{aligned}
f_1(y_1,\dots,y_N)&=0\\
&\vdots\\
f_m(y_1,\dots,y_N)&=0\\
y_1-a_{10}-\sum_{j=1}^{r} z_{1j}\,g_j&=0\\
&\vdots\\
y_N-a_{N0}-\sum_{j=1}^{r} z_{Nj}\,g_j&=0
\end{aligned}$$

This system has a solution in $\hat{R}$. Hence it has a solution in R which provides an approximation of the solution $\hat{a}$ of $f=0$ modulo $m^c\hat{R}$.

(6.11) <u>Remark</u>: By (6.10)(b) in order to show Theorem (6.8) it suffices to show that every system of equations over R which has a solution in $\hat{R}$ has a solution in R.

(6.12) <u>Lemma</u>: In order to prove (6.8) it suffices to show: If $f=(f_1,\dots,f_m)\subseteq (K[x_1,\dots,x_n])[y_1,\dots,y_N]$ so that $f(\hat{a})=0$ for some $\hat{a}=(\hat{a}_1,\dots,\hat{a}_N)\in\hat{R}^N=K[[x_1,\dots,x_n]]^N$, then there is an $a\in R^N$, $R=(K[x_1,\dots,x_n]_{(x_1,\dots,x_n)})^h$, so that $f(a)=0$.

<u>Proof</u>: Put $S=K[x_1,\dots,x_n]$, $n=(x_1,\dots,x_n)S$, and suppose that every system of equations over S which has a solution in $\hat{R}=K[[x_1,\dots,x_n]]$ has a solution in R.

Let $f=(f_1,\dots,f_m)\subseteq R[y_1,\dots,y_N]$ and $\hat{a}=(\hat{a}_1,\dots,\hat{a}_N)\in R^N$ with $f(\hat{a})=0$.

Consider the R-algebra morphism:

$$\varphi: R[y_1,\dots,y_N]\longrightarrow \hat{R}$$

defined by $\varphi(y_i)=\hat{a}_i$. Then $\ker(\varphi)=P$ is a prime ideal of $R[y_1,\dots,y_N]$ with $f=(f_1,\dots,f_m)\subseteq P$. We may assume that $f=P$. Let $P_0=P\cap S[y_1,\dots,y_N]$ be the contraction of P to $S[y_1,\dots,y_N]$. Since R is a direct limit of étale neighborhoods of S_n, the ring $R[y_1,\dots,y_N]$ is a direct limit of localized étale extensions of $S_n[y_1, -, y_N]$. Moreover, $R[y_1, -, y_N]/P_0R[y_1, -, y_N]$ is a direct limit of localized étale extensions of $S_n[y_1, -, y_N]/P_0$. In particular, $R[y_1, -, y_N]/P_0R[y_1, -, y_N]$ is a reduced ring.

If $P_0R[y_1, -, y_N] = P$ we are done. Hence assume $P_0R[y_1, -, y_N] \neq P$ and write:

$$P_0R[y_1, -, y_N] = P \cap Q$$

where $Q \nsubseteq P$ and Q is intersection of (finitely many) prime ideals which are not contained in P.

Let $P_0 = (h_1, -, h_t) \subseteq S[y_1, -, y_N]$ for some polynomials $h_i \in S[y_1, -, y_N]$. Since $\ker(\varphi) = P$ and $Q \nsubseteq P$ there is an $g \in Q$ with $g(\hat{a}) \neq 0$. Let $c \in \mathbb{N}$ with $g(\hat{a}) \in m^c - m^{c+1}$ and $c_{i0} \in S_n$ with $\hat{a}_i - c_{i0} \in m^{c+1}\hat{R}$. Write

$$c_{i0} = a_{i0}/b$$

where $a_{i0}, b \in S$ and $b \notin n$. Choose a system of generators $(g_1, -, g_r) = n^{c+1}$, $g_i \in S$, and consider the system of equations over S:

$$h_1(y_1, -, y_N) = 0$$
$$\vdots$$
$$h_t(y_1, -, y_N) = 0$$
$$a_{10} - by_1 - \sum_{j=1}^{r} g_j z_{1j} = 0$$
$$\vdots$$
$$a_{N0} - by_N - \sum_{j=1}^{r} g_j z_{Nj} = 0$$

where z_{ij}, $1 \leq i \leq N$, $1 \leq j \leq r$, are additional variables.

Since the system has a solution in $\hat{R}$, by assumption it has a solution $(a, d_{ij}) = (a_1, -, a_N, d_{ij}) \in R^{N(1+r)}$. We claim that $g(a) \neq 0$.

Since $a_{i0} - ba_i \in m^{c+1}$, $b\hat{a}_i - ba_i \in m^{c+1}$, and thus $\hat{a}_i - a_i \in m^{c+1}\hat{R}$, since b is invertible in R. Thus by Taylor's formula:

$$g(a) = g(\hat{a}) + \sum_{j=1}^{N} (\partial g/\partial y_j)(\hat{a})\,(a_j - \hat{a}_j) + \hat{u}$$

where $\hat{u}$ is a sum of multiples of $(a_j - \hat{a}_j)(a_k - \hat{a}_k)$. Hence $g(a) - g(\hat{a}) \in m^{c+1}\hat{R}$ and $g(a) \neq 0$, since $g(\hat{a}) \notin m^{c+1}\hat{R}$.

It remains to show that $f(a) = 0$ for all $f \in P$. Notice that $PQ \subseteq P_0R[y_1, -, y_N]$.

Hence for all $f \in P$ $fg \in P_0$ and thus $f(a)g(a)=0$. Since R is a domain, $f(a)=0$.

In order to continue with the proof of theorem (6.8) we need to use a property of affine rings which will be proved in the next chapter: Let K be a field and A a local K-algebra which is essentially of finite type over K. Then A and A^h are Nagata rings. This implies that if in addition A is a domain with field of quotients $L=Q(A)$, then for every minimal prime $Q \subseteq \widehat{A}$ the field $k(Q)=(\widehat{A}/Q)_Q$ is separable over L.

Let K be a field, $S=K[x_1,\ldots,x_n]$ the polynomial ring over K, $L=Q(S)$ its field of quotients, and $f=(f_1,\ldots,f_m) \subseteq S[y_1,\ldots,y_N]$ an ideal. If $\hat{a}=(\hat{a}_1,\ldots,\hat{a}_N) \in \widehat{R}^N = K[\![x_1,\ldots,x_n]\!]^N$ with $f(\hat{a})=0$ consider the S-algebra morphism

$$\varphi : S[y_1,\ldots,y_N] \longrightarrow \widehat{R}=K[\![x_1,\ldots,x_n]\!]$$

defined by $\varphi(y_i)=\hat{a}_i$. Then $P=\ker\varphi$ is a prime ideal of $S[y_1,\ldots,y_N]$ and there are field extensions:

$$L=Q(S) \subseteq E = Q(S[y_1,\ldots,y_N]/P) \subseteq Q(\widehat{R})=K((x_1,\ldots,x_n)).$$

Since $Q(\widehat{R})$ is separable over L, by (1.26) E is separable over L and by (2.20) E is smooth over L.

We may suppose that $f=(f_1,\ldots,f_m)=P$ with $ht\,P=N-r$. Since $P\cap S=0$, $\dim(L[y_1,\ldots,y_N]/PL[y_1,\ldots,y_N])=r$ and $(L[y_1,\ldots,y_N]/PL[y_1,\ldots,y_N])_{PL[y_1,\ldots,y_N]}=E$ is smooth over L. By (5.4) there is an $(N-r)\times(N-r)$ minor of the Jacobian matrix:

$$\left(\partial f_i/\partial y_j\right)_{\substack{1\le i\le m\\ 1\le j\le N}}$$

say

$$\delta = \det\left(\partial f_i/\partial y_j\right)_{1\le i,j\le N-r}$$

with $\delta\neq 0$ in $S[y_1,\ldots,y_N]/P$. Consider $\delta=\delta(x,y)\in S[y_1,\ldots,y_N]=K[x_1,\ldots,x_n,y_1,\ldots,y_N]$ as a polynomial in x_i and y_j and notice that

$$\delta(x,\hat{a})\neq 0 \quad \text{in } K[\![x_1,\ldots,x_n]\!]$$

since $P=\ker\varphi$.

(6.13) Lemma: It suffices to show:

For every $c \in \mathbb{N}$ there is an element $a = (a_1, \dots, a_N) \in R^N$ so that
$$f_1(a) = \dots = f_{N-r}(a) = 0$$
and $\hat{a}_i \equiv a_i \mod m^c \hat{R}$ for all $1 \le i \le N$.

Proof: By the proof of (5.4)(a) $\Rightarrow$ (b) we know that
$$P\,S[y_1, \dots, y_N]_P = (f_1, \dots, f_{N-r})\,S[y_1, \dots, y_N]_P .$$
This implies that
$$\mathrm{rad}\,(f_1, \dots, f_{N-r}) = P \cap Q$$
where Q is a finite intersection of prime ideals not containing P or $Q = S[y_1, \dots, y_N]$.
Let $g \in Q - P$, then $g(x, \hat{a}) \neq 0$ since $P = \ker \varphi$. Assume that $g(x,\hat{a}) \in \hat{m}^d - \hat{m}^{d+1}$ and let $c \in \mathbb{N}$ with $c > d$. We claim that if $a = (a_1, \dots, a_N) \in R^N$ with
$$f_1(a) = \dots = f_{N-r}(a) = 0$$
and $\hat{a}_i \equiv a_i \mod m^c R$ for all $1 \le i \le N$, then $h(a) = 0$ for all $h \in P$.
By Taylor's formula:
$$g(x,a) = g(x,\hat{a}) + \sum_{j=1}^{N} (\partial g/\partial y_j)(\hat{a})(a_j - \hat{a}_j) + \hat{u}$$
where $\hat{u}$ is the sum of multiples of $(a_i - \hat{a}_i)(a_j - \hat{a}_j)$. Hence $g(x,a) - g(x,\hat{a}) \in m^c \hat{R}$ with $g(x,\hat{a}) \notin m^c \hat{R}$. Thus $g(x,a) \neq 0$. Since $PQ \subseteq \mathrm{rad}\,(f_1, \dots, f_{N-r})$, for all $h \in P$ there is a $t \in \mathbb{N}$ with $(gh)^t(a) = 0$ and thus $h(a) = 0$, since R is a domain.

This shows that it suffices to prove the following statement:

$$(*)\left\{\begin{array}{l} \text{Let } f_1, \dots, f_m \in S[y_1, \dots, y_N] \text{ be polynomials in } x_i \text{ and } y_j \text{ with } m \le N \text{ and} \\ \hat{a} = (\hat{a}_1, \dots, \hat{a}_N) \in \hat{R}^N = K[\![x_1, \dots, x_n]\!]^N \text{ an element with the following} \\ \text{properties:} \\ \text{(a) } f_i(\hat{a}) = 0 \text{ for all } 1 \le i \le m \\ \text{(b) The Jacobian } \delta(x,y) = \det\left(\partial f_i/\partial y_j\right)_{1 \le i,j \le m} \text{ satisfies that} \\ \qquad\qquad \delta(x,\hat{a}) \neq 0 \end{array}\right.$$

Then we have to show:

$$(**)\begin{cases} \text{For all } c\in\mathbb{N} \text{ there is an element } a=(a_1,\dots,a_N)\in R^N \text{ with} \\ \text{(i)}\quad f_i(a)=0 \quad \text{for all } 1\le i\le m \\ \text{(ii)}\quad \hat{a}_j \equiv a_j \bmod m^c\hat{R} \quad \text{for all } 1\le j\le N. \end{cases}$$

Recall Tougeron's theorem (5.9):

Let (R,m,k) be a local Noetherian Henselian ring, $f_1,\dots,f_m\in R[y_1,\dots,y_N]$, and

$$\Delta = \left(\partial f_i/\partial y_j\right)_{\substack{1\le i\le m\\ 1\le j\le N}}$$

the Jacobian matrix of $f_1,\dots,f_m$. For a fixed element $a=(a_1,\dots,a_N)\in R^N$ consider the R-linear map:

$$\varphi:\ R^N \xrightarrow{\Delta(a)} R^m$$

defined by multiplication by $\Delta(a)$. Let $C=R^m/\operatorname{im}\varphi$ and $I=\operatorname{ann}_R(C)$.

Then Tougeron's theorem states:

If there is an $a=(a_1,\dots,a_N)\in R^N$ so that

$$f_i(a)\equiv 0 \bmod I^2m^d$$

for all $1\le i\le m$ and some $d\in\mathbb{N}$, $d>0$, then there is a $b=(b_1,\dots,b_N)\in R^N$ with

$$f_i(b)=0 \quad \text{for all } 1\le i\le m$$

$$\text{and}\quad b_j\equiv a_j \bmod Im^d \quad \text{for all } 1\le j\le N.$$

Let $f_1,\dots,f_m\in S[y_1,\dots,y_N]$ with $m\le N$,

$$\Delta = \left(\partial f_i/\partial y_j\right)_{\substack{1\le i\le m\\ 1\le j\le N}}$$

the Jacobian matrix, and

$$\delta=\delta(x,y) = \det\left(\partial f_i/\partial y_j\right)_{1\le i,j\le m}.$$

Let $R=\left(K[x_1,\dots,x_n]_{(x_1,\dots,x_n)}\right)^h$, $a=(a_1,\dots,a_N)\in R^N$ and

$$\varphi:\ R^N \xrightarrow{\Delta(a)} R^m$$

multiplication by $\Delta(a)$. With $C = R^m/\operatorname{im}\varphi$ and $I = \operatorname{ann}_R(C)$ we claim:

(6.14) <u>Lemma:</u> $\delta(x,a) \in I$

<u>Proof:</u> Consider the $m \times N$ Jacobian matrix

$$\Delta = (\partial f_i/\partial y_j)_{\substack{1\le i\le m\\ 1\le j\le N}} = \begin{bmatrix} \partial f_1/\partial y_1 & \cdots\cdots & \partial f_1/\partial y_N \\ \vdots & & \vdots \\ \partial f_m/\partial y_1 & \cdots\cdots & \partial f_m/\partial y_N \end{bmatrix}$$

Then there is an $N \times m$-matrix Γ so that:

$$\Delta\Gamma = \begin{bmatrix} \delta & & 0 \\ & \ddots & \\ 0 & & \delta \end{bmatrix}.$$

Let e_j be the j^{th} canonical basis vector of R^N and Γ_j the j^{th} column of Γ. Then

$$\delta(x,a)e_j = \Delta(a)\Gamma(a)e_j = \Delta(a)\Gamma_j(a)$$

and $\delta(x,a)e_j \in \operatorname{im}\varphi$ for all $1 \le j \le N$. Thus $\delta(x,a) \in I$.

Therefore in order to prove (**) by (5.9) (Tougeron) it suffices to show:

$$(\ast\ast\ast)\begin{cases} \text{For every } c \in \mathbb{N} \text{ there is an } a^0 = (a_1^0, \ldots, a_N^0) \in R^N \text{ so that} \\ (\alpha)\quad f_i(a^0) \equiv 0 \mod \delta^2(x,a^0)\,m^c \qquad \text{for all } 1 \le i \le m \\ (\beta)\quad \hat{a}_j \equiv a_j^0 \mod m^c\hat{R} \qquad \text{for all } 1 \le j \le N. \end{cases}$$

(6.15) <u>Lemma:</u> Let $n \ge 1$ and suppose that Theorem (6.8) has been shown for equations over $R_0 = (K[x_1, \ldots, x_k]_{(x_1,\ldots,x_k)})^h$ where $0 \le k < n$. Let $R = (K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)})^h$, $g, f_1, \ldots, f_m \in K[x_1, \ldots, x_n, y_1, \ldots, y_N]$ polynomials, and $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N) \in \hat{R}^N = K[\![x_1, \ldots, x_n]\!]^N$ with

$$g(x,\hat{a}) \neq 0$$
$$f_i(x,\hat{a}) = 0 \quad \text{for all } 1 \le i \le m.$$

Then for every $c \in \mathbb{N}$ there is an element $a = (a_1, \ldots, a_N) \in R^N$ so that:

$$f_i(x,a) \equiv 0 \mod (g(x,a)) \quad \text{and} \quad \hat{a}_j \equiv a_j \mod (m^c\hat{R}) \quad \text{for all } 1 \le i \le m;\ 1 \le j \le N.$$

(6.16) <u>Lemma</u>: Lemma (6.15) implies Theorem (6.8).

<u>Proof</u>: We have to show that under the assumptions of (*) conditions ($^{**}_{*}$) follow.
Let $f_1, \ldots, f_m \in S[y_1, \ldots, y_N] = K[x_1, \ldots, x_n, y_1, \ldots, y_N]$ with $m \leq N$,

$$\delta = \delta(x,y) = \det \left(\partial f_i / \partial y_j\right)_{1 \leq i,j \leq m}$$

the Jacobian, and $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N) \in \hat{R}^N$ with

(a) $f_i(x, \hat{a}) = 0$ for all $1 \leq i \leq m$

(b) $\delta(x, \hat{a}) \neq 0$

We want to show ($^{**}_{*}$), that is,

For all $c \in \mathbb{N}$ there is an $a^0 = (a_1^0, \ldots, a_N^0) \in R^N$ so that

(α) $f_i(a^0) \equiv 0 \mod \delta^2(x, a^0)\, m^c$ for all $1 \leq i \leq m$

(β) $\hat{a}_j \equiv a_j^0 \mod m^c \hat{R}$ for all $1 \leq j \leq N$.

We can assume that $\hat{a} \neq (0, \ldots, 0)$. Hence after renumbering – if necessary – set

$$g(x,y) = \delta^2(x,y)\, x_1^c$$

to obtain

$$f_i(x, \hat{a}) = 0 \quad \text{for all } 1 \leq i \leq m$$

and $\quad g(x, \hat{a}) \neq 0$.

By Lemma (6.15) for every $c \in \mathbb{N}$ there is an element $a^0 = (a_1^0, \ldots, a_N^0) \in R^N$ with

$$f_i(x, a^0) \equiv 0 \mod (g(x, a^0)) \quad \text{for all } 1 \leq i \leq m$$

$$\hat{a}_j \equiv a_j^0 \mod (m^c \hat{R}).$$

Thus conditions ($^{**}_{*}$) are satisfied and the theorem follows.

<u>Proof of</u> (6.15): Let $g, f_1, \ldots, f_m \in K[x_1, \ldots, x_n, y_1, \ldots, y_N]$ and $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_N) \in \hat{R}^N$ with:

$$g(x, \hat{a}) \neq 0$$

$$f_i(x, \hat{a}) = 0 \quad \text{for all } 1 \leq i \leq m.$$

Note that for all $\hat{b} \in \hat{R}^N$ with $\hat{a}_j \equiv \hat{b}_j \mod m^c \hat{R}$ for all $1 \leq j \leq N$,

$$g(x, \hat{a}) \equiv g(x, \hat{b}) \mod m^c \hat{R}.$$

In particular, if $g(x, \hat{a})$ is invertible then $g(x, \hat{b})$ is invertible provided that

$\hat{a} \equiv \hat{b} \mod m^c \hat{R}^N$ and $c \geq 1$.

By the Weierstraß preparation theorem there are integers $e_i \in \mathbb{N}$ so that with

$$t_i = x_i + x_n^{e_i} \quad \text{for } 1 \leq i \leq n-1$$

$$t_n = x_n$$

$g(x,\hat{a}) = g(t,\hat{a})$ can be written as:

$$g(t,\hat{a}) = \varepsilon\, \hat{h}(t_n)$$

where $\varepsilon \in K[\![x_1, \ldots, x_n]\!]$ invertible and

$$\hat{h}(t_n) = t_n^r + \hat{h}_{r-1} t_n^{r-1} + \ldots + \hat{h}_0$$

where for all $0 \leq i \leq r-1$

$$\hat{h}_i \in K[\![t_1, \ldots, t_{n-1}]\!].$$

Replacing t by x we may assume that

$$g(x,\hat{a}) = \varepsilon\, \hat{h}(x_n)$$

where $\varepsilon \in K[\![x_1, \ldots, x_n]\!]$ is a unit and $h(x_n)$ is a monic polynomial in x_n of degree r over the power series ring $K[\![x_1, \ldots, x_{n-1}]\!]$.

Obviously, the ring

$$\hat{B} = K[\![x_1, \ldots, x_n]\!]/(g(x,\hat{a})) = K[\![x_1, \ldots, x_n]\!]/(h(x_n))$$

is a finite extension of the power series ring $K[\![x_1, \ldots, x_{n-1}]\!]$. Let $z_{\nu j}$, $1 \leq \nu \leq N$, $0 \leq j \leq r-1$ be variables and put:

$$z_\nu^* = \sum_{j=0}^{r-1} z_{\nu j}\, x_n^j \quad \text{for } 1 \leq \nu \leq N.$$

Substitute z_ν^* for y_ν and consider $g, f_1, \ldots, f_m$ as polynomials in $K[x_i, z_{\nu j}]$ where $1 \leq \nu \leq N$, $1 \leq i \leq n$, and $0 \leq j \leq r-1$. Finally introduce variables $w_0, \ldots, w_{r-1}$ and write

$$w(x_n) = x_n^r + w_{r-1} x_n^{r-1} + \ldots\ldots + w_1 x_1 + w_0.$$

Since $w(x_n)$ is monic in x_n we have the following equations in the polynomial ring $K[x_i, z_{\nu j}, w_\mu]$:

$$g(x, z^*) = w(x_n)\, Q + \sum_{j=0}^{r-1} G_j\, x_n^j$$

$$f_i(x, z^*) = w(x_n)\, Q_i + \sum_{j=0}^{r-1} F_{ij}\, x_n^j \quad \text{for all } 1 \leq i \leq m$$

where

$$Q, Q_i \in K[x_1, -, x_n, z_{\nu j}, w_\mu]$$
$$G_j, F_{ij} \in K[x_1, -, x_{n-1}, z_{\nu j}, w_\mu].$$

We want to investigate a system of equations over the polynomial ring $K[x_1, -, x_{n-1}]$ which relates to the old system ($f_i = 0$) via the polynomials G_j and F_{ij}. First note that for all $1 \le \nu \le n$:

$$\hat{a}_\nu = \hat{h}(x_n)\,\hat{q}_\nu + \sum_{j=0}^{r-1} \hat{a}_{\nu j}\, x_n^j$$

where $\hat{q}_\nu \in K[[x_1, -, x_n]]$ and $\hat{a}_{\nu j} \in K[[x_1, -, x_{n-1}]]$ for all ν and j. Let

$$\hat{a}_\nu^* = \sum_{j=0}^{r-1} a_{\nu j}\, x_n^j.$$

Then for all $1 \le \nu \le N$:

$$\hat{a}_\nu \equiv \hat{a}_\nu^* \mod (\hat{h}(x_n))$$

and hence by Taylor:

$$g(x, \hat{a}^*) \equiv g(x, \hat{a}) \mod (\hat{h}(x_n))$$
$$f_i(x, \hat{a}^*) \equiv f_i(x, \hat{a}) \mod (\hat{h}(x_n)) \quad \text{for } 1 \le i \le m.$$

Substituting $\hat{a}_{\nu j}$ for $z_{\nu j}$ and $\hat{h}_\mu$ for w_μ we have

$$G_j(x_1, -, x_{n-1}, \hat{a}_{\nu j}, \hat{h}_\mu) = 0 \quad \text{for } 0 \le j \le r-1$$
$$F_{ik}(x_1, -, x_{n-1}, \hat{a}_{\nu j}, \hat{h}_\mu) = 0 \quad \text{for } 1 \le i \le m, 0 \le k \le r-1.$$

This yields a system of equations over the polynomial ring $K[x_1, -, x_{n-1}]$:

$$G_j(x_1, -, x_{n-1}, z_{\nu j}, w_\mu) = 0 \quad , \; 0 \le j \le r-1$$
$$F_{ik}(x_1, -, x_{n-1}, z_{\nu j}, w_\mu) = 0 \quad , \; 1 \le i \le m, \; 0 \le k \le r-1.$$

By induction hypothesis this system is solvable in $R_0 = (K[x_1, -, x_{n-1}]_{(x_1, -, x_{n-1})})^h$ with solutions being arbitrarily close to $\hat{a}_{\nu j}$ and $\hat{h}_\mu$. Let $c \in \mathbb{N}$ and $a_{\nu j}, h_\mu \in R_0$ with

$$G_i(x_1, -, x_{n-1}, a_{\nu j}, h_\mu) = 0 \quad \text{for all } 0 \le i \le r-1$$
$$F_{ik}(x_1, -, x_{n-1}, a_{\nu j}, h_\mu) = 0 \quad \text{for all } 1 \le i \le m, 0 \le k \le r-1$$

and
$$\hat{a}_{\nu j} \equiv a_{\nu j} \mod m_0^c R_0$$
$$\hat{h}_\mu \equiv h_\mu \mod m_0^c R_0 \quad \text{for all } \nu, j, \mu,$$

where $m_0 = (x_1, -, x_{n-1}) R_0$ is the maximal ideal of R_0. Let $q_\nu \in R$ so that

$$\hat{q}_\nu \equiv q_\nu \mod m^c R$$

and set: $$h(x_n) = x_n^r + h_{r-1} x_n^{r-1} + \dots + h_1 x_n + h_0$$
$$a_\nu^* = \sum_{j=0}^{r-1} a_{\nu j} x_n^j$$
$$a_\nu = h(x_n) q_\nu + a_\nu^*$$

Then:

$$a_\nu \equiv a_\nu^* \mod (h(x_n))$$
$$g(x, a^*) \equiv 0 \mod (h(x_n))$$
$$f_i(x, a^*) \equiv 0 \mod (h(x_n))$$

and therefore by Taylor:

$$g(x,a) \equiv 0 \mod (h(x_n))$$
$$f_i(x,a) \equiv 0 \mod (h(x_n)) \qquad \text{for all } 1 \le i \le m.$$

Let $c > r$, then $g(x,a)$ is regular in x_n since $g(x,\hat{a})$ contains an unmixed term x_n^r. Moreover, x_n^r is the smallest unmixed term in $g(x,a)$ (and $g(x,\hat{a})$). Therefore $g(x,a) = \gamma\, h(x_n)$ where $\gamma \in K[\![x_1, \dots, x_n]\!]$ invertible. Hence

$$f_i(x,a) \equiv 0 \mod (g(x,a)) \quad \text{for all } 1 \le i \le m$$
$$g(x,a) \neq 0$$
$$\hat{a}_\nu \equiv a_\nu \mod m^c R$$

for $c \in \mathbb{N}$ sufficiently large. This shows the Lemma.

(6.17) <u>Corollary</u>: Let K be a field, $R = (K[x_1, \dots, x_n]_{(x_1, \dots, x_n)})^h$, and $I \subseteq R$ an ideal. The ring $T = R/I$ has the approximation property.

<u>Proof</u>: Let $I = (h_1, \dots, h_t) \subseteq R$ and $f = (f_1, \dots, f_m) \subseteq T[y_1, \dots, y_N]$ polynomials over T. Suppose that there is an element $\hat{a} = (\hat{a}_1, \dots, \hat{a}_N) \in \hat{T}^N$ with $f_i(\hat{a}) = 0$ for all $1 \le i \le m$. Choose elements

$$F = (F_1, \dots, F_m) \subseteq R[y_1, \dots, y_N] \quad \text{and}$$
$$\hat{b} = (\hat{b}_1, \dots, \hat{b}_N) \in \hat{R}^N$$

so that $F_i + IR[y_1, \dots, y_N] = f_i$ for $1 \le i \le m$ and $\hat{b}_j + I\hat{R} = \hat{a}_j$ for $1 \le j \le N$. Then $F_i(\hat{b}) \in I\hat{R}$ for all $1 \le i \le m$ and there are elements $\hat{c}_{ij} \in \hat{R}$ with

$$F_i(\hat{b}) = \sum_{j=1}^{t} \hat{c}_{ij} h_j .$$

Let z_{ij}, $1 \le i \le m$, $1 \le j \le t$ be additional variables. The system

$$(*) \begin{cases} F_1(y) - \sum_{j=1}^{t} z_{1j} h_j = 0 \\ \quad \vdots \\ F_m(y) - \sum_{j=1}^{t} z_{mj} h_j = 0 \end{cases}$$

has the solution $(\hat{b}, \hat{c})$ in $\hat{R}$. Hence by (6.8) $(*)$ has a solution in R which yields a solution of the system $f_i = 0$ in T.

The next three results on rings with approximation property are mentioned without proofs.

(6.18) Theorem: (M. Artin) There is a function $\beta: \mathbb{N}^4 \longrightarrow \mathbb{N}$ with the following property: Let K be a field, $f = (f_1, -, f_m) \subseteq K[x_1, -, x_n, y_1, -, y_N]$ polynomials over K of total degree $\deg(f_i) \le d$ for all $1 \le i \le m$. Suppose that there is an element $\bar{a} = (\bar{a}_1, -, \bar{a}_N) \in K[x_1, -, x_n]^N$ so that

$$f_i(x, \bar{a}) \equiv 0 \mod (x_1, -, x_n)^{\beta} \quad \text{for all } 1 \le i \le m,$$

where $\beta = \beta(n, N, d, \alpha)$. Then there is an element $a = (a_1, -, a_N) \in R^N$, where $R = (K[x_1, -, x_n]_{(x_1, -, x_n)})^h$, with

$$f_i(x, a) = 0 \quad \text{for all } 1 \le i \le m \text{ and}$$
$$a_j \equiv \bar{a}_j \mod (x_1, -, x_n)^{\alpha} \quad \text{for all } 1 \le j \le N.$$

(6.19) Corollary: Let K be a field and $f = (f_1, -, f_m) \subseteq K[x_1, -, x_n, y_1, -, y_N]$ polynomials over K. Suppose that for all $\beta \in \mathbb{N}$ there is an element $\bar{a}_\beta = (\bar{a}_{\beta 1}, -, \bar{a}_{\beta N}) \in K[x_1, -, x_n]^N$ with

$$f_i(x, \bar{a}_\beta) \equiv 0 \mod (x_1, -, x_n)^{\beta} \quad \text{for all } 1 \le i \le m .$$

Then the system $f_i = 0$ has a solution in $R = (K[x_1, -, x_n]_{(x_1, -, x_n)})^h$.

(6.20) Definition: A local Noetherian ring (R, m) is said to have the strong approximation property if R satisfies the following condition: Let $f=(f_1,\ldots,f_t) \subseteq R[Y_1,\ldots,Y_N]$ be polynomials over R so that for all $c \in \mathbb{N}$ there is an element $a_c \in R^N$ with $f_i(a_c) \equiv 0 \mod m^c$ for all $1 \le i \le t$. Then there is an element $a \in R^N$ with $f_i(a) = 0$ for all $1 \le i \le t$.

(6.21) Theorem: (M. Artin and others) A complete local Noetherian ring $(\hat{R}, m)$ has the strong approximation property.

(6.22) Corollary: A local Noetherian ring (R,m) with the approximation property has the strong approximation property.

As an application of the approximation property (and the strong approximation property) we show:

(6.23) Theorem: (M. Hochster) Let (R,m) be a local Noetherian factorial ring with approximation property. Then $\hat{R}$ is factorial.

Proof: If R is a local Noetherian ring with approximation property then R is excellent. In particular, the completion of a local normal excellent ring is normal. Hence $\hat{R}$ is a local normal ring and a domain. Since in a Noetherian domain every nonunit is a finite product of irreducible elements, it remains to show that every irreducible element $\hat{p}$ of $\hat{R}$ is a prime element. Suppose that there are elements $\hat{f}, \hat{g} \in \hat{R}$ with $\hat{p} \mid \hat{f}\hat{g}$. Thus there is an $\hat{h} \in \hat{R}$ with $\hat{h}\hat{p} = \hat{f}\hat{g}$. Consider the equation $HP = FG$ over R where H, P, F, G are variables. Since R has the approximation property, for all $n \in \mathbb{N}$ there are elements $h_n, p_n, f_n, g_n \in R$ with

$$h_n p_n = f_n g_n$$

and $\hat{h} \equiv h_n,\ \hat{p} \equiv p_n,\ \hat{f} \equiv f_n,\ \hat{g} \equiv g_n \mod \hat{m}^n$.

If the elements $p_n \in R$ are reducible for infinitely many n (i.e. $p_n = u_n v_n$ where $u_n, v_n \in R$ and neither u_n nor v_n a unit), then, since $\widehat{R}$ has the strong approximation property, $\widehat{p}$ is reducible in $\widehat{R}$. Thus there is an $n_0 \in \mathbb{N}$ with p_n a prime element of R for all $n \in \mathbb{N}$ with $n \geq n_0$. Hence there are infinitely many $n \in \mathbb{N}$ with $p_n \mid f_n$ or $p_n \mid g_n$. Assume that $p_n \mid f_n$ for infinitely many $n \in \mathbb{N}$. Then by the strong approximation property $\widehat{p}$ divides $\widehat{f}$.